Math Challenge 2. Find the exact value of $a > 0$ that maximizes the area between the graph of

$$f(x) = x^a(1 - x^a)$$

and the $x-$axis from $x = 0$ to $x = 1$. Use some test to verify that you actually have the maximum area.

Solution. The area under the graph of f from $x = 0$ to 1 is

$$A(a) = \int_0^1 f(x)\,dx = \int_0^1 \left(x^a - x^{2a}\right)\,dx = \left[\frac{x^{a+1}}{a+1} - \frac{x^{2a+1}}{2a+1}\right]\Bigg|_0^1 = \frac{1}{a+1} - \frac{1}{2a+1}.$$

Here we find the value of a that maximizes $A(a)$ over the interval $(0, \infty)$.

We compute $\dfrac{dA}{da} = -\dfrac{1}{(a+1)^2} + \dfrac{2}{(2a+1)^2}$ and then solve $\dfrac{dA}{da} = 0$ for a.

$$\begin{aligned} \frac{dA}{da} = 0 \;&\Leftrightarrow\; \frac{2}{(2a+1)^2} = \frac{1}{(a+1)^2} \\ &\Leftrightarrow\; 2(a+1)^2 = (2a+1)^2 \\ &\Leftrightarrow\; 2a^2 + 4a + 2 = 4a^2 + 4a + 1 \\ &\Leftrightarrow\; 1 = 2a^2 \;\Leftrightarrow\; a = \frac{1}{\sqrt{2}} \quad \text{(because } a > 0\text{).} \end{aligned}$$

Note that $A(a)$ is continuous and differentiable on $(0, \infty)$ and that $A'(a) > 0$ when $0 < a < 1/\sqrt{2}$ and $A'(a) < 0$ when $a > 1/\sqrt{2}$. Thus, by the Increasing/Decreasing Test, A is increasing on $(0, 1/\sqrt{2})$ and decreasing on $(1/\sqrt{2}, \infty)$.

Therefore, $A(a)$ attains its absoulte maximum when $a = 1/\sqrt{2}$.