Math Challenge 3. Without the use of a computer or calculator, find the exact value of the sum

$$S = \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+2019}.$$

Solution.

$$\begin{aligned}
S &= \sum_{n=1}^{2019} \frac{1}{1+2+3+\ldots+n} \\
&= \sum_{n=1}^{2019} \frac{1}{\frac{n(n+1)}{2}} \\
&= 2\sum_{n=1}^{2019} \frac{1}{n(n+1)} \\
&= 2\sum_{n=1}^{2019} \left(\frac{1}{n} - \frac{1}{n+1}\right) \qquad \text{(partial fractions)} \\
&= 2\left[1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \ldots + \cancel{\frac{1}{2018}} - \cancel{\frac{1}{2019}} + \cancel{\frac{1}{2019}} - \frac{1}{2020}\right] \\
&= 2\left[1 - \frac{1}{2020}\right] \qquad \text{(telescoping sum)} \\
&= \frac{2019}{1010}.
\end{aligned}$$