Math Challenge 4. A small country has only two coin denominations, 5¢ coins and 7¢ coins. They are proud of the fact that every value greater than 23 can be written as a nonnegative linear combination of these two coin values. For example,

$$29¢ = 2(7¢) + 3(5¢).$$

Prove that every value greater than 23 can be written as a nonnegative linear combination of these two coin values.

Solution. We use mathematical induction on $n \geq 24$ to prove the claim.

(i) Basis step: Note that $24 = 2(7¢) + 2(5¢)$, which means 24 is a nonnegative linear combination of 7 and 5.

(ii) Inductive step: Let $k \geq 24$, and suppose that k can be written as a nonnegative linear combination of 7 and 5. Then $k = r(7¢) + s(5¢)$ for some nonnegative integers r and s. We will show that $k+1$ can also be written as a nonnegative linear combination of 7 and 5.

Here, we may assume $0 \leq r \leq 4$, because when $r \geq 5$, we may let $r \equiv r_0 \pmod 5$ and rewrite $k = (r_0 + r - r_0)(7¢) + s(5¢) = r_0(7¢) + (r - r_0)(7¢) + s(5¢) = r_0(7¢) + 5t(7¢) + s(5¢) = r_0(7¢) + (7t + s)(5¢)$ for some positive integer t. So, we have indeed five cases on r.

(Case 1) $r = 0$. Then $k = s(5¢)$ with $s \geq 5$. But then

$$k + 1 = s(5¢) + 1 = s(5¢) + 21 - 20 = 3(7¢) + (s - 4)(5¢).$$

(Case 2) $r = 1$. Then $k = (7¢) + s(5¢)$ with $s \geq 4$. But then

$$k + 1 = (7¢) + s(5¢) + 1 = (7¢) + s(5¢) + 21 - 20 = 4(7¢) + (s - 4)(5¢).$$

(Case 3) $r = 2$. Then $k = 2(7¢) + s(5¢)$ with $s \geq 2$. But then

$$k + 1 = 2(7¢) + s(5¢) + 1 = 2(7¢) + s(5¢) - 14 + 15 = 0(7¢) + (s + 3)(5¢).$$

(Case 4) $r = 3$. Then $k = 3(7¢) + s(5¢)$ with $s \geq 1$. But then

$$k + 1 = 3(7¢) + s(5¢) + 1 = 3(7¢) + s(5¢) - 14 + 15 = 1(7¢) + (s + 3)(5¢).$$

(Case 5) $r = 4$. Then $k = 4(7¢) + s(5¢)$ with $s \geq 0$. But then

$$k + 1 = 4(7¢) + s(5¢) + 1 = 4(7¢) + s(5¢) - 14 + 15 = 2(7¢) + (s + 3)(5¢).$$

In each case, $k + 1$ is a nonnegative linear combination of 7 and 5.

Therefore, by mathematical induction, every value $n \geq 24$ can be written as a nonnegative linear combination of 7 and 5.