Math Challenge 5. Evaluate $\sqrt{7-\cfrac{1}{7-\cfrac{1}{7-\ddots}}}$.

Express your answer in the form $\dfrac{a+b\sqrt{c}}{d}$ where $a, b, c, d \in \mathbb{Z}$.

Solution. The nested fraction under the big radical notation is the limit of the sequence that can be recursively defined by

$$x_1 = 7, \quad x_{n+1} = 7 - \frac{1}{x_n} \quad \text{for} \quad n \geq 1.$$

By using mathematical induction, one can show that the sequence (x_n) is a monotone decreasing sequence bounded below by 6. Thus, (x_n) is convergent. Let $x = \lim_{n\to\infty} x_n$. Then by the recurrence relation above, x satisfies the equation

$$x = 7 - \frac{1}{x} \quad \Leftrightarrow \quad x^2 - 7x + 1 = 0.$$

Then using the quadratic formula, we obtain $x = \dfrac{7 \pm 3\sqrt{5}}{2}$. Because the sequence (x_n) is bounded below by 6, we must have $x \geq 6$, i.e., $x = \dfrac{7+3\sqrt{5}}{2}$.

The solution to the problem is then

$$\sqrt{x} = \sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{\sqrt{7+3\sqrt{5}}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{14+6\sqrt{5}}}{2}$$

$$= \frac{\sqrt{9+6\sqrt{5}+5}}{2} = \frac{\sqrt{\left(3+\sqrt{5}\,\right)^2}}{2} = \frac{3+\sqrt{5}}{2}$$

as requested.

A general formula provided by Dr. Bangteng Xu:

$$\sqrt{n-\cfrac{1}{n-\cfrac{1}{n-\ddots}}} = \frac{\sqrt{n+2} + \sqrt{n-2}}{2}$$