Math Challenge 6. Billy has 20 identical pieces of candy. His mother says he should share them with his 5 friends. How many different ways can he distribute the candy if some might get no candy or someone might get all the candy?

Solution. We use the stars and bars method in which stars represent the candy and bars represent the separators placed between Billy's five friends. Thus, we have 20 stars and 4 bars. Note that each combination of candy distribution can be represented by a 5-tuple $(f_1, f_2, f_3, f_4, f_5)$, where f_i stands for the number of candy that the i^{th} friend receives. Here are some example distributions:

★★ | ★★★ | ★★★★★★★ | ★★ | ★★★★★★ is equivalent to $(2, 3, 7, 2, 6)$,
★★★★★★★★★★★ | | ★★★★★★★★ | | ★ is equivalent to $(11, 0, 8, 0, 1)$,
| | | ★★★★★★★★★★★★★★★★★★★★ | is equivalent to $(0, 0, 0, 20, 0)$.

In each of the combination above, there are 24 places, and the problem is where to place the 4 bars. With this in mind,

★★ | ★★★ | ★★★★★★★ | ★★ | ★★★★★★ is equivalent to $\{3, 7, 15, 18\}$,
★★★★★★★★★★★ | | ★★★★★★★★ | | ★ is equivalent to $\{12, 13, 22, 23\}$,
| | | ★★★★★★★★★★★★★★★★★★★★ | is equivalent to $\{1, 2, 3, 24\}$.

Therefore, the problem boils down to finding the number of 4-element subsets of the set $\{1, 2, 3, \dots, 23, 24\}$, which is equal to

$$\binom{24}{4} = \frac{24!}{4!\,20!} = \frac{24 \cdot 23 \cdot 22 \cdot 21}{4 \cdot 3 \cdot 2 \cdot 1} = 10{,}626.$$