Math Challenge 9. Without the use of a computer or calculator, evaluate the definite integral

$$\int_1^{2019} \frac{\sqrt{x}}{\sqrt{2020-x}+\sqrt{x}}\,dx.$$

Solution. Set $u = 2020 - x$. Then $du = -dx$ and hence the integral becomes

$$\begin{aligned} I &= \int_{2019}^{1} \frac{\sqrt{2020-u}}{\sqrt{u}+\sqrt{2020-u}}\,(-du) \\ &= \int_1^{2019} \frac{\sqrt{2020-u}}{\sqrt{2020-u}+\sqrt{u}}\,du = \int_1^{2019} \frac{\sqrt{2020-x}}{\sqrt{2020-x}+\sqrt{x}}\,dx. \end{aligned}$$

Thus,

$$\begin{aligned} I + I &= \int_1^{2019} \frac{\sqrt{x}}{\sqrt{2020-x}+\sqrt{x}}\,dx + \int_1^{2019} \frac{\sqrt{2020-x}}{\sqrt{2020-x}+\sqrt{x}}\,dx \\ &= \int_1^{2019} \left(\frac{\sqrt{x}}{\sqrt{2020-x}+\sqrt{x}} + \frac{\sqrt{2020-x}}{\sqrt{2020-x}+\sqrt{x}} \right) dx \\ &= \int_1^{2019} 1\,dx \\ &= 2018. \end{aligned}$$

Therefore, $2I = 2018$ implying $I = 1009$.