Math Challenge 10. Without the use of a computer or calculator, find the exact value of

$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\ldots+\sqrt{1+\frac{1}{2019^2}+\frac{1}{2020^2}}.$$

Solution. Each term in the sum is the square root of the quantity

$$1+\frac{1}{n^2}+\frac{1}{(n+1)^2} = \frac{n^2(n+1)^2+(n+1)^2+n^2}{n^2(n+1)^2}.$$

Note that the numerator can be manipulated to a perfect square as follows.

$$\begin{aligned}
n^2(n+1)^2+(n+1)^2+n^2 &= (n+1)^2\left(n^2+1\right)+n^2 \\
&= (n+1)^2\left(n^2+2n+1-2n\right)+n^2 \\
&= (n+1)^2\left((n+1)^2-2n\right)+n^2 \\
&= (n+1)^4-2n(n+1)^2+n^2 \\
&= \left((n+1)^2-n\right)^2.
\end{aligned}$$

Then the sum is equal to

$$\begin{aligned}
\sum_{n=1}^{2019}\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}} &= \sum_{n=1}^{2019}\frac{(n+1)^2-n}{n(n+1)} = \sum_{n=1}^{2019}\left(\frac{n+1}{n}-\frac{1}{n+1}\right) \\
&= \sum_{n=1}^{2019}1+\sum_{n=1}^{2019}\left(\frac{1}{n}-\frac{1}{n+1}\right) \\
&= 2019+1-\frac{1}{2020} \quad \text{(the second sum is telescoping)} \\
&= \frac{2019(2021)}{2020}.
\end{aligned}$$