Math Challenge 12. Let $f(x) = \dfrac{3^x}{3^x + \sqrt{3}}$. Without the use of a computer or calculator, find the exact value of the sum

$$f\left(\frac{1}{2021}\right) + f\left(\frac{2}{2021}\right) + f\left(\frac{3}{2021}\right) + \ldots + f\left(\frac{2020}{2021}\right).$$

Solution. Let S denote the sum. Write it in both the given order and the reversed order.

$$S = f\left(\frac{1}{2021}\right) + f\left(\frac{2}{2021}\right) + \ldots + f\left(\frac{2020}{2021}\right)$$

$$S = f\left(\frac{2020}{2021}\right) + f\left(\frac{2019}{2021}\right) + \ldots + f\left(\frac{1}{2021}\right)$$

Adding the corresponding terms by grouping, we obtain

$$2S = \left[f\left(\frac{1}{2021}\right) + f\left(\frac{2020}{2021}\right)\right] + \left[f\left(\frac{2}{2021}\right) + f\left(\frac{2019}{2021}\right)\right] + \ldots + \left[f\left(\frac{2020}{2021}\right) + f\left(\frac{1}{2021}\right)\right].$$

Note that there are 2020 grouped terms in this sum and each group is a number in the form

$$f(t) + f(1-t) = \frac{3^t}{3^t + \sqrt{3}} + \frac{3^{1-t}}{3^{1-t} + \sqrt{3}}$$

$$= \frac{3^t\left(3^{1-t} + \sqrt{3}\right) + 3^{1-t}\left(3^t + \sqrt{3}\right)}{\left(3^t + \sqrt{3}\right)\left(3^{1-t} + \sqrt{3}\right)} = \frac{3 + 3^t\sqrt{3} + 3 + 3^{1-t}\sqrt{3}}{3 + 3^t\sqrt{3} + 3^{1-t}\sqrt{3} + 3} = 1.$$

Thus, $2S = 2020(1)$ and $\boxed{S = 1010}$.

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