Math Challenge 13. The number 4 can be expressed as an ordered sum of two or more positive integers in seven ways:

$$3 + 1, \quad 1 + 3, \quad 2 + 2,$$

$$2 + 1 + 1, \quad 1 + 2 + 1, \quad 1 + 1 + 2, \quad 1 + 1 + 1 + 1.$$

In how many ways can 20 be so expressed?

Solution. We use the stars and bars method in which stars represent ones (1), and bars represent plus signs (+) placed between the added numbers. Then the position(s) of bars (+) will show that each of the arrangements above is indeed equivalent to a unique set. Note that when there are 4 stars, there are 3 places that the bars can be put, and we choose a subset of those three places to put our bars, except we never put no bars, so we won't use the empty set.

Sum	Stars-bars Representation	Equivalent Set (Positions of bars)
$3 + 1$	★★★|★	$\{3\}$
$1 + 3$	★|★★★	$\{1\}$
$2 + 2$	★★|★★	$\{2\}$
$2 + 1 + 1$	★★|★|★	$\{2, 3\}$
$1 + 2 + 1$	★|★★|★	$\{1, 3\}$
$1 + 1 + 2$	★|★|★★	$\{1, 2\}$
$1 + 1 + 1 + 1$	★|★|★|★	$\{1, 2, 3\}$

When the number in question is 4, the number of ways to express it as an ordered sum of two or more positive integers is equal to the number of nonempty subsets of $\{1, 2, 3\}$, namely, $2^3 - 1 = 7$. (Here, we use the fact that the number of subsets of an n-element subset is 2^n.)

Similarly, the number of ways to express 20 as an ordered sum of two or more positive integers is equal to the number of nonempty subsets of $\{1, 2, 3, 4, \dots, 19\}$, which is $2^{19} - 1 = 524{,}287$.