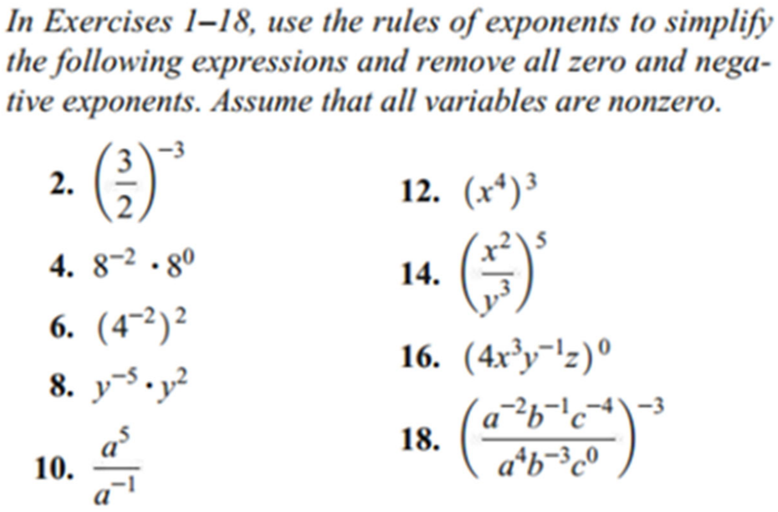
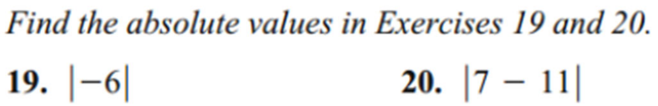
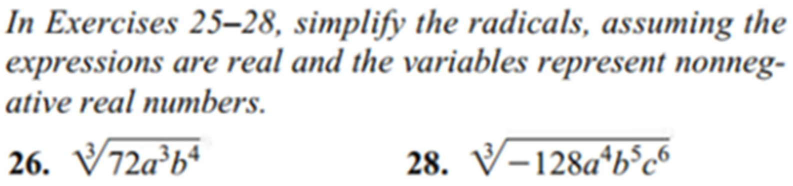
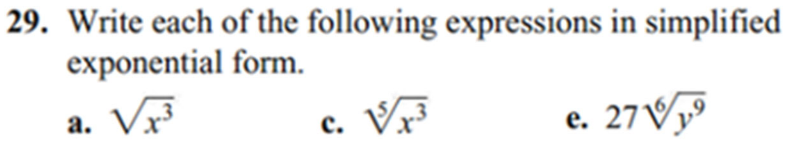
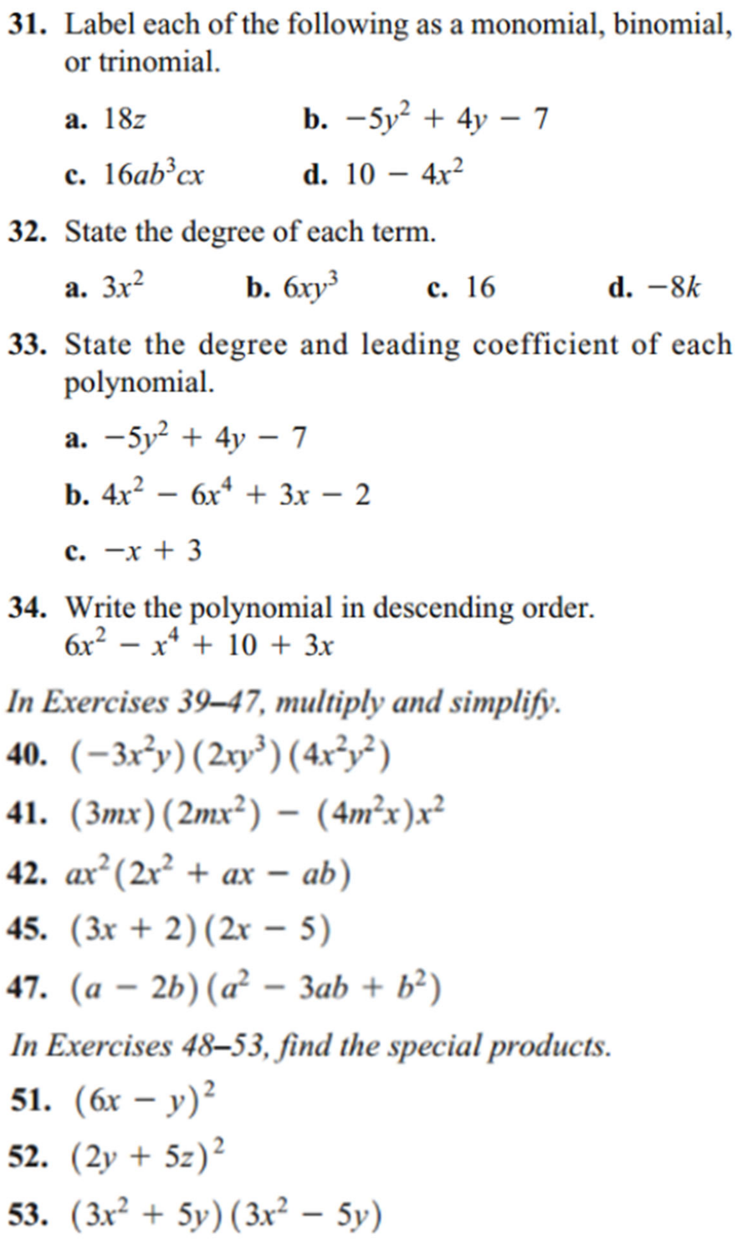
MAT 112A Final Exam Review Packet

Chapter 3: Quadratic, Piecewise-Defined, and Power Functions

Chapter 3 Algebra Toolbox

Learning Outcomes and Notes	**Examples**
Simplify expressions containing exponents.	 *In Exercises 1–18, use the rules of exponents to simplify the following expressions and remove all zero and negative exponents. Assume that all variables are nonzero.* **2.** $\left(\frac{3}{2}\right)^{-3}$ **4.** $8^{-2} \cdot 8^0$ **6.** $(4^{-2})^2$ **8.** $y^{-5} \cdot y^2$ **10.** $\frac{a^5}{a^{-1}}$ **12.** $(x^4)^3$ **14.** $\left(\frac{x^2}{y^3}\right)^5$ **16.** $(4x^3y^{-1}z)^0$ **18.** $\left(\frac{a^{-2}b^{-1}c^{-4}}{a^4b^{-3}c^0}\right)^{-3}$
Evaluate expressions containing absolute values.	 *Find the absolute values in Exercises 19 and 20.* **19.** $\lvert -6 \rvert$ **20.** $\lvert 7 - 11 \rvert$
Evaluate expressions containing radicals. Simplify expressions containing radicals.	 *In Exercises 25–28, simplify the radicals, assuming the expressions are real and the variables represent nonnegative real numbers.* **26.** $\sqrt[3]{72a^3b^4}$ **28.** $\sqrt[3]{-128a^4b^5c^6}$
Rewrite radical expressions using rational exponents.	 **29.** Write each of the following expressions in simplified exponential form. **a.** $\sqrt{x^3}$ **c.** $\sqrt[5]{x^3}$ **e.** $27\sqrt[6]{y^9}$

Identify characteristics of polynomials.



31. Label each of the following as a monomial, binomial, or trinomial.

a. $18z$ **b.** $-5y^2 + 4y - 7$

c. $16ab^3cx$ **d.** $10 - 4x^2$

32. State the degree of each term.

a. $3x^2$ **b.** $6xy^3$ **c.** 16 **d.** $-8k$

33. State the degree and leading coefficient of each polynomial.

a. $-5y^2 + 4y - 7$

b. $4x^2 - 6x^4 + 3x - 2$

c. $-x + 3$

34. Write the polynomial in descending order.
$6x^2 - x^4 + 10 + 3x$

In Exercises 39–47, multiply and simplify.

40. $(-3x^2y)(2xy^3)(4x^2y^2)$

41. $(3mx)(2mx^2) - (4m^2x)x^2$

42. $ax^2(2x^2 + ax - ab)$

45. $(3x + 2)(2x - 5)$

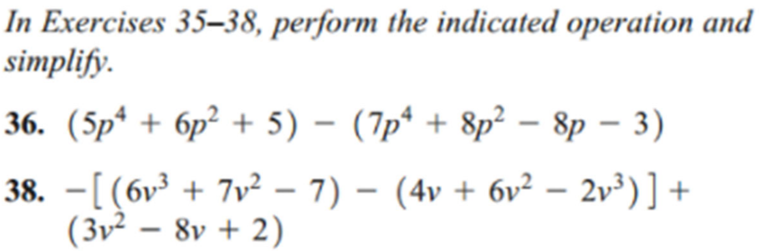
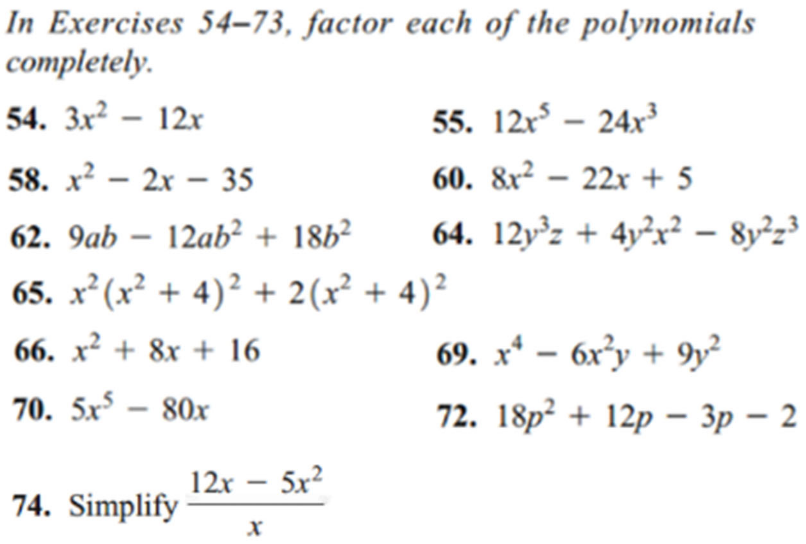
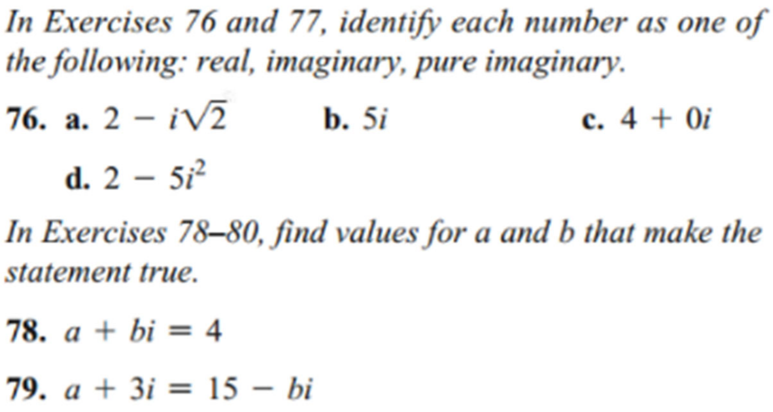
47. $(a - 2b)(a^2 - 3ab + b^2)$

In Exercises 48–53, find the special products.

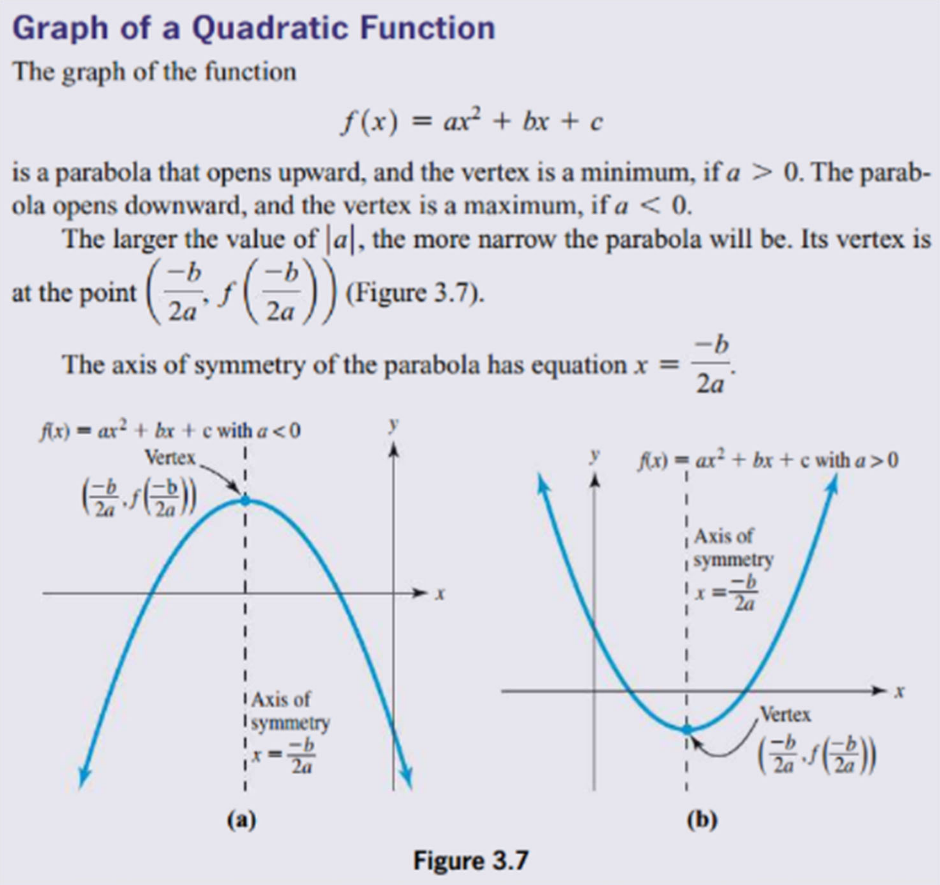
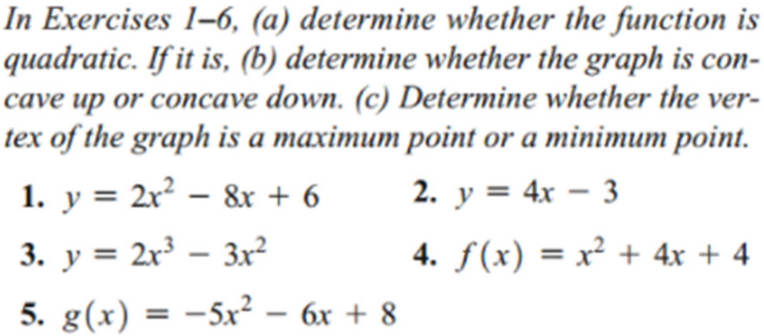
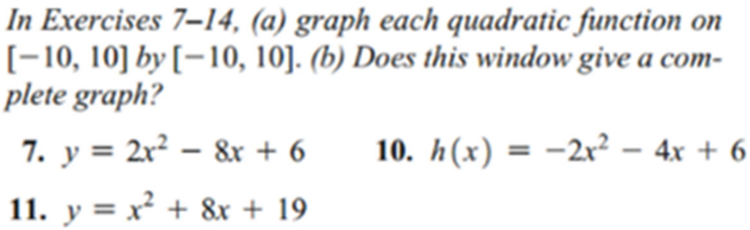
51. $(6x - y)^2$

52. $(2y + 5z)^2$

53. $(3x^2 + 5y)(3x^2 - 5y)$

Perform operations on polynomials.	 *In Exercises 35–38, perform the indicated operation and simplify.* **36.** $(5p^4 + 6p^2 + 5) - (7p^4 + 8p^2 - 8p - 3)$ **38.** $-[(6v^3 + 7v^2 - 7) - (4v + 6v^2 - 2v^3)] + (3v^2 - 8v + 2)$
Factor polynomials.	 *In Exercises 54–73, factor each of the polynomials completely.* **54.** $3x^2 - 12x$ **55.** $12x^5 - 24x^3$ **58.** $x^2 - 2x - 35$ **60.** $8x^2 - 22x + 5$ **62.** $9ab - 12ab^2 + 18b^2$ **64.** $12y^3z + 4y^2x^2 - 8y^2z^3$ **65.** $x^2(x^2 + 4)^2 + 2(x^2 + 4)^2$ **66.** $x^2 + 8x + 16$ **69.** $x^4 - 6x^2y + 9y^2$ **70.** $5x^5 - 80x$ **72.** $18p^2 + 12p - 3p - 2$ **74.** Simplify $\frac{12x - 5x^2}{x}$
Answer questions involving complex numbers.	 *In Exercises 76 and 77, identify each number as one of the following: real, imaginary, pure imaginary.* **76.** **a.** $2 - i\sqrt{2}$ **b.** $5i$ **c.** $4 + 0i$ **d.** $2 - 5i^2$ *In Exercises 78–80, find values for a and b that make the statement true.* **78.** $a + bi = 4$ **79.** $a + 3i = 15 - bi$

3.1 Quadratic Equations; Parabolas

Learning Outcomes and Notes	**Examples**
Describe characteristics of the graph of a quadratic function.  **Graph of a Quadratic Function** The graph of the function $f(x) = ax^2 + bx + c$ is a parabola that opens upward, and the vertex is a minimum, if $a > 0$. The parabola opens downward, and the vertex is a maximum, if $a < 0$. The larger the value of $\lvert a \rvert$, the more narrow the parabola will be. Its vertex is at the point $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ (Figure 3.7). The axis of symmetry of the parabola has equation $x = \frac{-b}{2a}$. (a) (b) Figure 3.7	 *In Exercises 1–6, (a) determine whether the function is quadratic. If it is, (b) determine whether the graph is concave up or concave down. (c) Determine whether the vertex of the graph is a maximum point or a minimum point.* **1.** $y = 2x^2 - 8x + 6$ **2.** $y = 4x - 3$ **3.** $y = 2x^3 - 3x^2$ **4.** $f(x) = x^2 + 4x + 4$ **5.** $g(x) = -5x^2 - 6x + 8$
Graph quadratic functions in a given viewing window.	 *In Exercises 7–14, (a) graph each quadratic function on* $[-10, 10]$ *by* $[-10, 10]$. *(b) Does this window give a complete graph?* **7.** $y = 2x^2 - 8x + 6$ **10.** $h(x) = -2x^2 - 4x + 6$ **11.** $y = x^2 + 8x + 19$

Write the equation of a quadratic function given a graph or points on the graph.

Graph of a Quadratic Function

In general, the graph of the function

$$y = a(x - h)^2 + k$$

is a parabola with its vertex at the point (h, k).

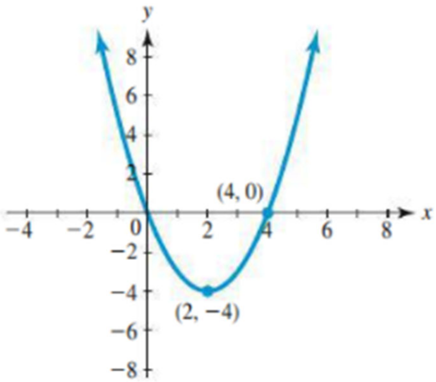
The parabola opens upward if $a > 0$, and the vertex is a minimum.

The parabola opens downward if $a < 0$, and the vertex is a maximum.

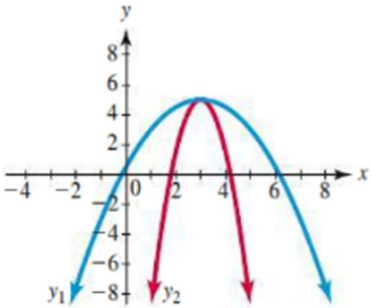
The axis of symmetry of the parabola has equation $x = h$.

The a is the same as the leading coefficient in $y = ax^2 + bx + c$, so the larger the value of $|a|$, the narrower the parabola will be.

15. Write the equation of the quadratic function whose graph is shown.

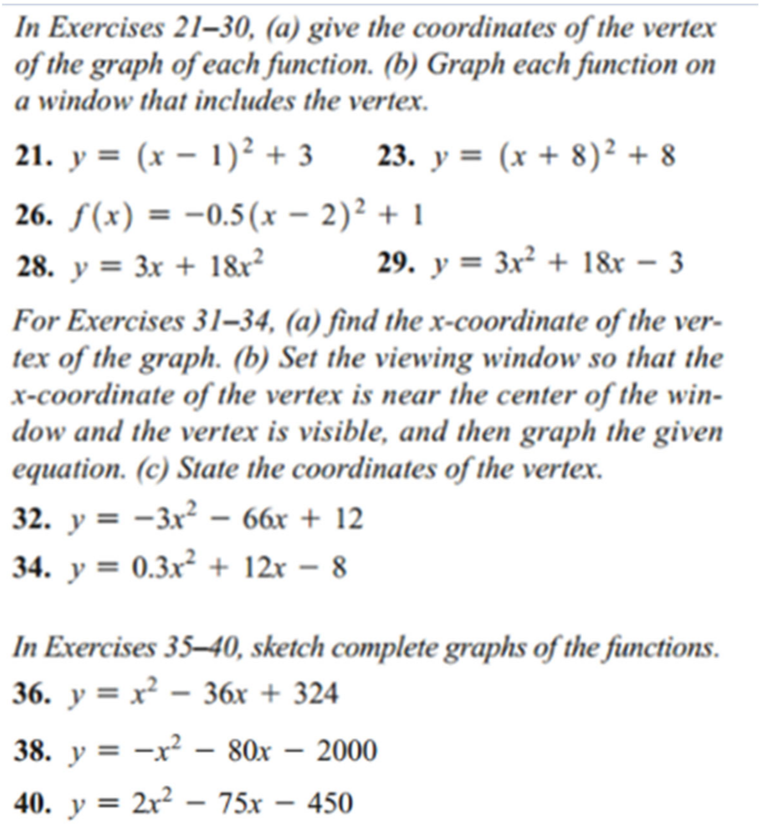
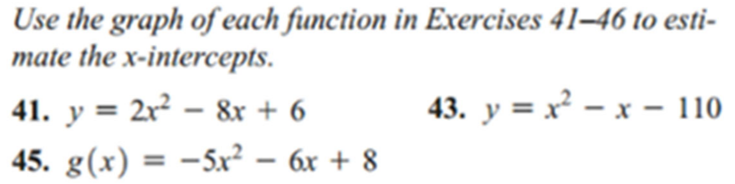


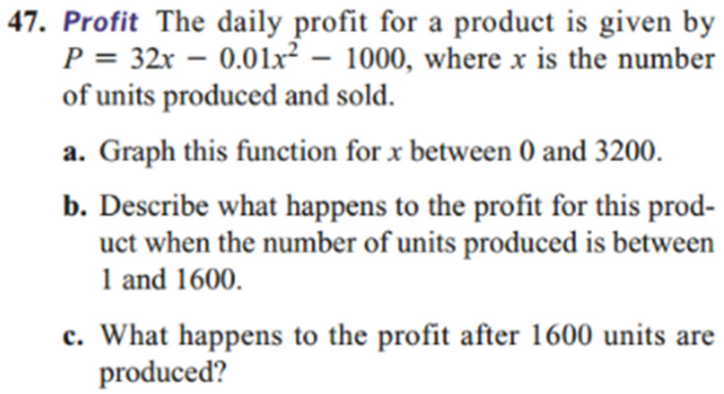
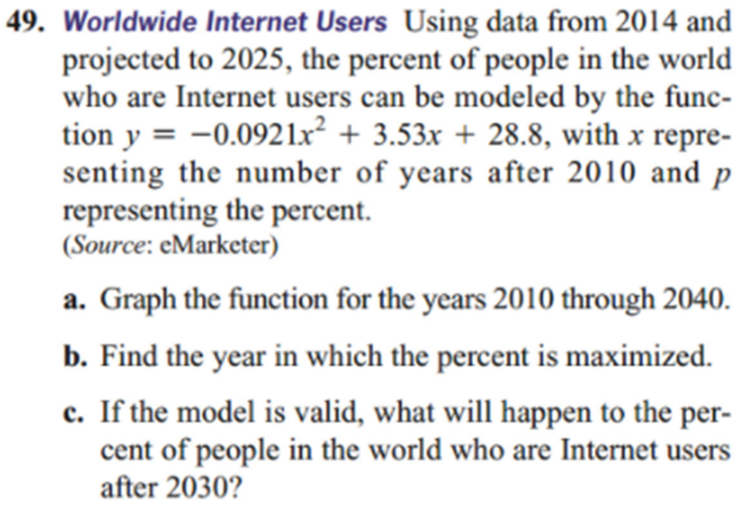
18. The two graphs shown have equations of the form $y = -a(x - 3)^2 + 5$. Is the value of $|a|$ larger for y_1 or y_2?

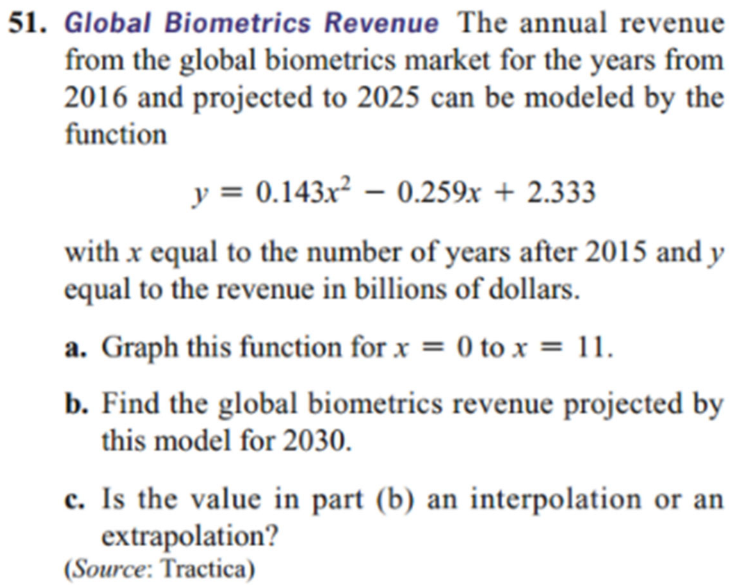
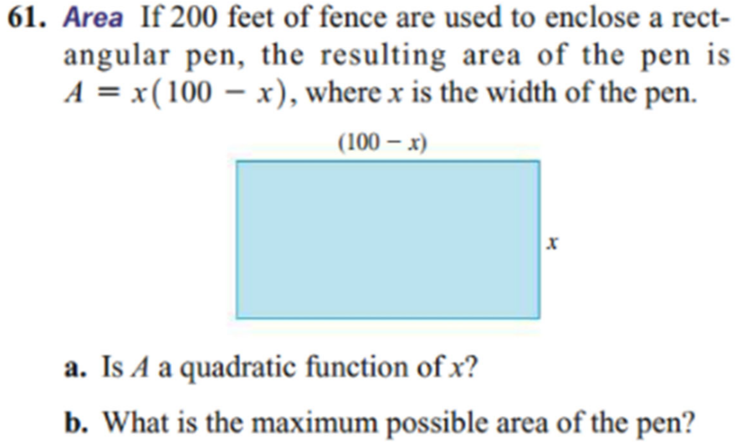


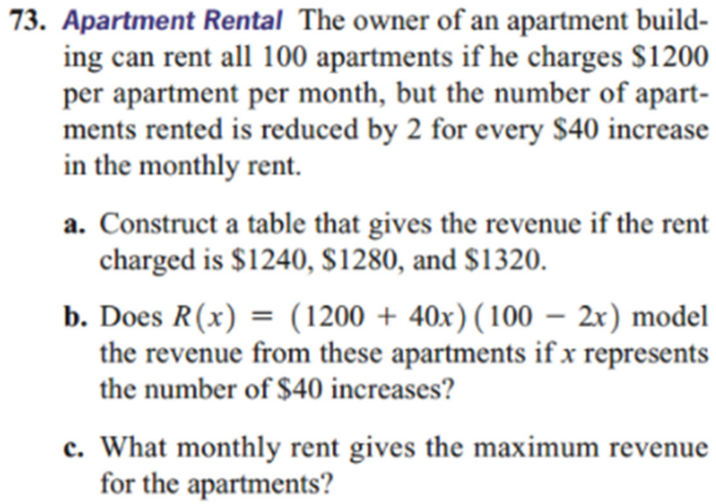
19. If the points in the table lie on a parabola, write the equation whose graph is the parabola.

x	−1	1	3	5
y	−7	13	−7	−67

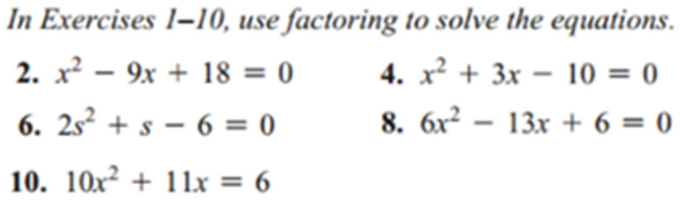
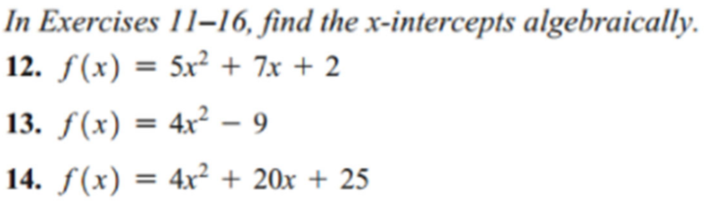
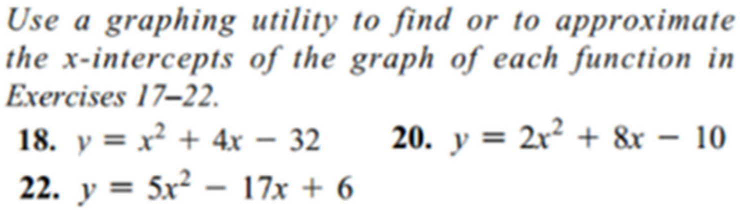
Find the coordinates of the vertex and graph quadratic functions.	 *In Exercises 21–30, (a) give the coordinates of the vertex of the graph of each function. (b) Graph each function on a window that includes the vertex.* **21.** $y = (x - 1)^2 + 3$ **23.** $y = (x + 8)^2 + 8$ **26.** $f(x) = -0.5(x - 2)^2 + 1$ **28.** $y = 3x + 18x^2$ **29.** $y = 3x^2 + 18x - 3$ *For Exercises 31–34, (a) find the x-coordinate of the vertex of the graph. (b) Set the viewing window so that the x-coordinate of the vertex is near the center of the window and the vertex is visible, and then graph the given equation. (c) State the coordinates of the vertex.* **32.** $y = -3x^2 - 66x + 12$ **34.** $y = 0.3x^2 + 12x - 8$ *In Exercises 35–40, sketch complete graphs of the functions.* **36.** $y = x^2 - 36x + 324$ **38.** $y = -x^2 - 80x - 2000$ **40.** $y = 2x^2 - 75x - 450$
Use the graph of a quadratic function to estimate the *x*-intercepts.	 *Use the graph of each function in Exercises 41–46 to estimate the x-intercepts.* **41.** $y = 2x^2 - 8x + 6$ **43.** $y = x^2 - x - 110$ **45.** $g(x) = -5x^2 - 6x + 8$

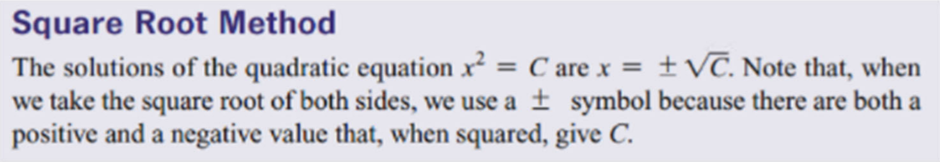
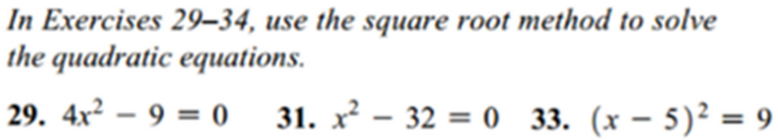
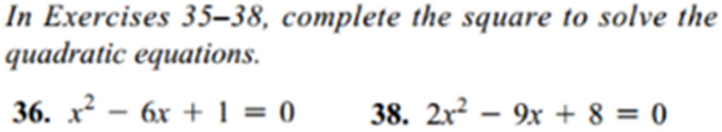
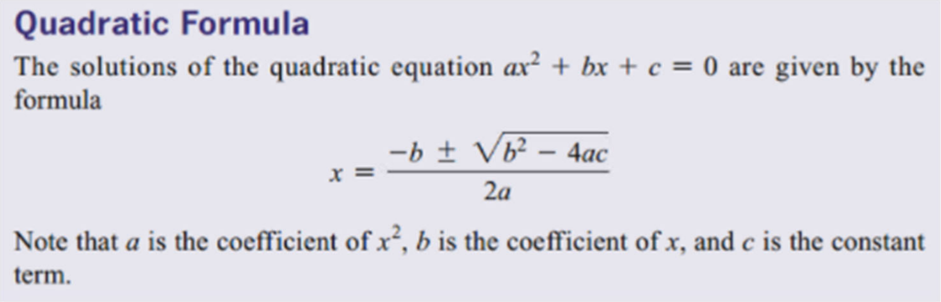
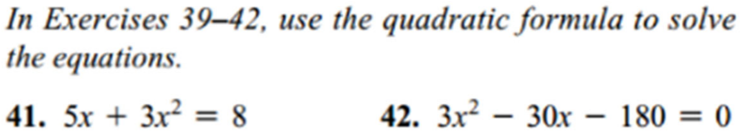
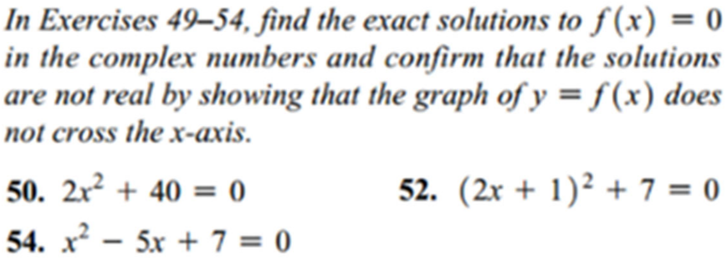
Solve applications involving quadratic functions.	 **47.** **Profit** The daily profit for a product is given by $P = 32x - 0.01x^2 - 1000$, where x is the number of units produced and sold. **a.** Graph this function for x between 0 and 3200. **b.** Describe what happens to the profit for this product when the number of units produced is between 1 and 1600. **c.** What happens to the profit after 1600 units are produced?
Answer interactive figure questions involving increasing/decreasing functions.	 **49.** **Worldwide Internet Users** Using data from 2014 and projected to 2025, the percent of people in the world who are Internet users can be modeled by the function $y = -0.0921x^2 + 3.53x + 28.8$, with x representing the number of years after 2010 and p representing the percent. (*Source*: eMarketer) **a.** Graph the function for the years 2010 through 2040. **b.** Find the year in which the percent is maximized. **c.** If the model is valid, what will happen to the percent of people in the world who are Internet users after 2030?

	 **51.** ***Global Biometrics Revenue*** The annual revenue from the global biometrics market for the years from 2016 and projected to 2025 can be modeled by the function $y = 0.143x^2 - 0.259x + 2.333$ with x equal to the number of years after 2015 and y equal to the revenue in billions of dollars. **a.** Graph this function for $x = 0$ to $x = 11$. **b.** Find the global biometrics revenue projected by this model for 2030. **c.** Is the value in part (b) an interpolation or an extrapolation? (*Source*: Tractica)
Answer interactive figure questions involving quadratic function graphs.	 **61.** ***Area*** If 200 feet of fence are used to enclose a rectangular pen, the resulting area of the pen is $A = x(100 - x)$, where x is the width of the pen. **a.** Is A a quadratic function of x? **b.** What is the maximum possible area of the pen?

	**73.** *Apartment Rental* The owner of an apartment building can rent all 100 apartments if he charges \$1200 per apartment per month, but the number of apartments rented is reduced by 2 for every \$40 increase in the monthly rent. **a.** Construct a table that gives the revenue if the rent charged is \$1240, \$1280, and \$1320. **b.** Does $R(x) = (1200 + 40x)(100 - 2x)$ model the revenue from these apartments if x represents the number of \$40 increases? **c.** What monthly rent gives the maximum revenue for the apartments?

3.2 Solving Quadratic Equations

Learning Outcomes and Notes	**Examples**
Solve quadratic equations using factoring. ***See the handout on the AC Method!***	*In Exercises 1–10, use factoring to solve the equations.* **2.** $x^2 - 9x + 18 = 0$ **4.** $x^2 + 3x - 10 = 0$ **6.** $2s^2 + s - 6 = 0$ **8.** $6x^2 - 13x + 6 = 0$ **10.** $10x^2 + 11x = 6$
Find x-intercepts algebraically.	*In Exercises 11–16, find the x-intercepts algebraically.* **12.** $f(x) = 5x^2 + 7x + 2$ **13.** $f(x) = 4x^2 - 9$ **14.** $f(x) = 4x^2 + 20x + 25$
Solve quadratic equations and find x-intercepts graphically.	*Use a graphing utility to find or to approximate the x-intercepts of the graph of each function in Exercises 17–22.* **18.** $y = x^2 + 4x - 32$ **20.** $y = 2x^2 + 8x - 10$ **22.** $y = 5x^2 - 17x + 6$

Solve quadratic equations using the square root method.  **Square Root Method** The solutions of the quadratic equation $x^2 = C$ are $x = \pm\sqrt{C}$. Note that, when we take the square root of both sides, we use a ± symbol because there are both a positive and a negative value that, when squared, give C.	 *In Exercises 29–34, use the square root method to solve the quadratic equations.* **29.** $4x^2 - 9 = 0$ **31.** $x^2 - 32 = 0$ **33.** $(x - 5)^2 = 9$
Solve quadratic equations by completing the square.	 *In Exercises 35–38, complete the square to solve the quadratic equations.* **36.** $x^2 - 6x + 1 = 0$ **38.** $2x^2 - 9x + 8 = 0$
Solve quadratic equations using the quadratic formula.  **Quadratic Formula** The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Note that a is the coefficient of x^2, b is the coefficient of x, and c is the constant term.	 *In Exercises 39–42, use the quadratic formula to solve the equations.* **41.** $5x + 3x^2 = 8$ **42.** $3x^2 - 30x - 180 = 0$
Solve quadratic equations having complex solutions.	 *In Exercises 49–54, find the exact solutions to $f(x) = 0$ in the complex numbers and confirm that the solutions are not real by showing that the graph of $y = f(x)$ does not cross the x-axis.* **50.** $2x^2 + 40 = 0$ **52.** $(2x + 1)^2 + 7 = 0$ **54.** $x^2 - 5x + 7 = 0$

Use the discriminant to determine the number of solutions of a quadratic equation.

The Discriminant

We can also determine the type of solutions a quadratic equation has by looking at the expression $b^2 - 4ac$, which is called the **discriminant** of the quadratic equation $ax^2 + bx + c = 0$. The discriminant is the expression inside the radical in the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so it determines if the quantity inside the radical is positive, zero, or negative.

- If $b^2 - 4ac > 0$, there are two different real solutions.
- If $b^2 - 4ac = 0$, there is one real solution.
- If $b^2 - 4ac < 0$, there is no real solution.

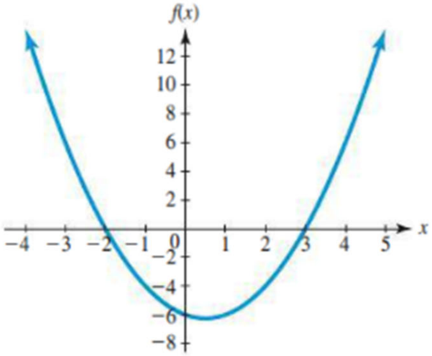
In Exercises 55–58, you are given the graphs of several functions of the form $f(x) = ax^2 + bx + c$ for different values of a, b, and c. For each function,

a. Determine whether the discriminant is positive, negative, or zero.

b. Determine whether there are 0, 1, or 2 real solutions to $f(x) = 0$.

c. Solve the equation $f(x) = 0$, if possible.

55.



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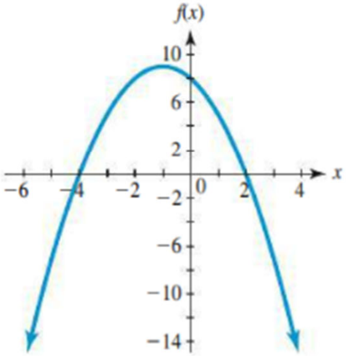
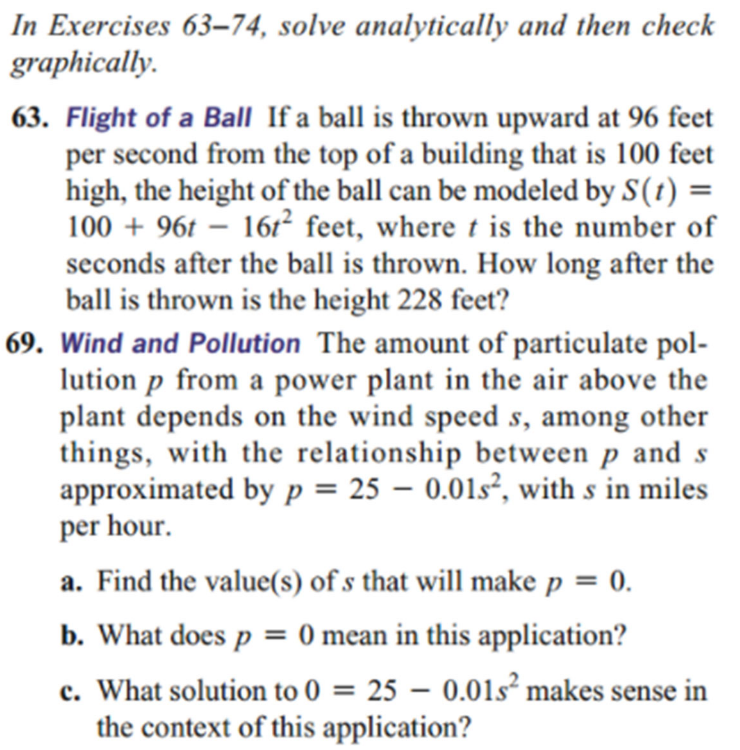


For each function in Exercise 59–62,

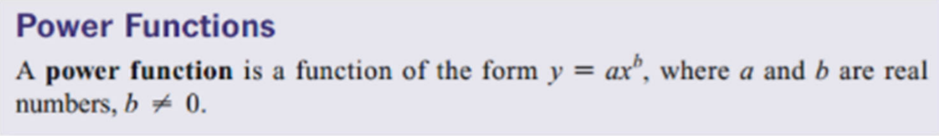
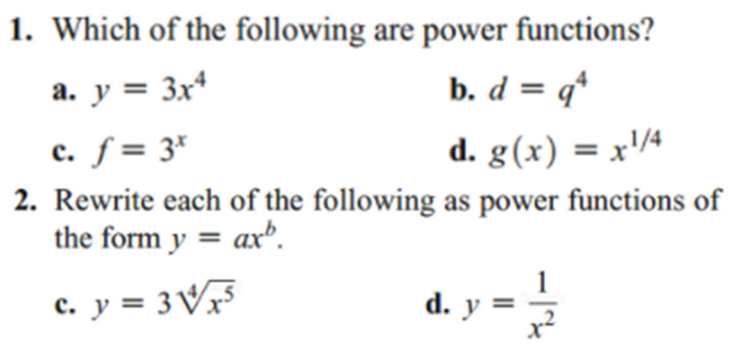
a. Calculate the discriminant.

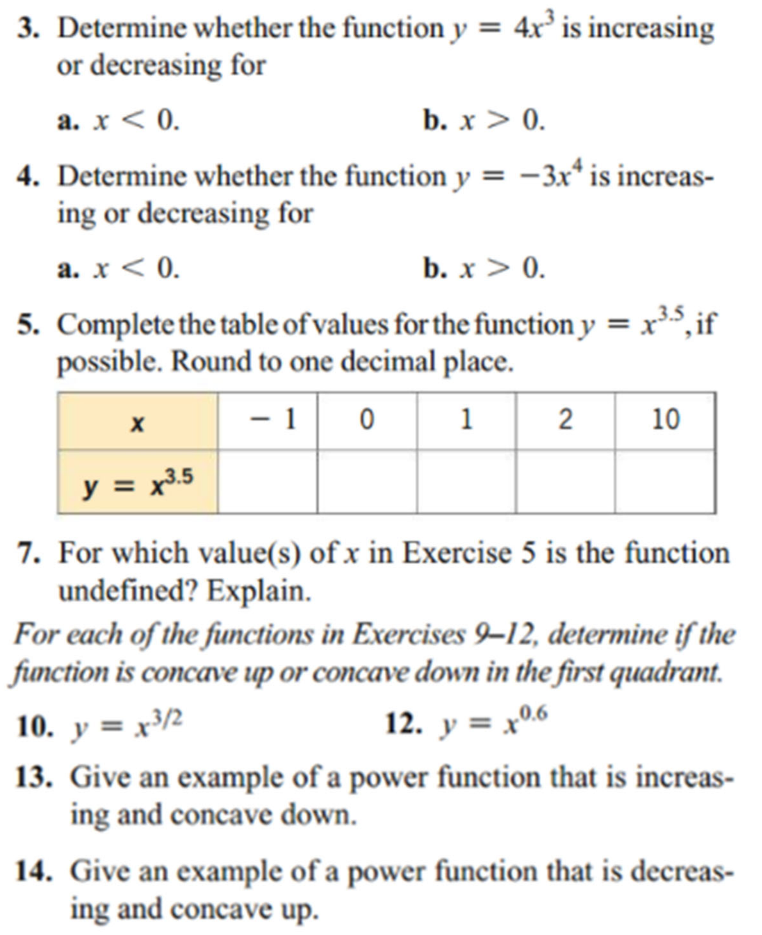
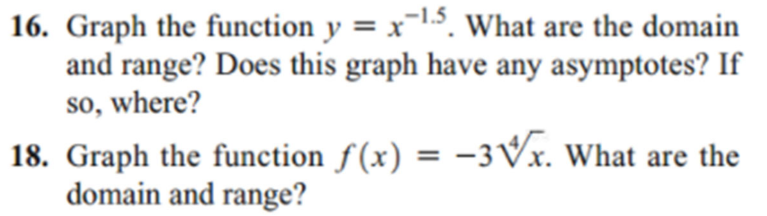
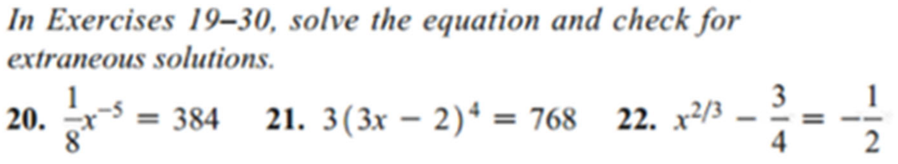
b. Determine whether there are 0, 1, or 2 real solutions to $f(x) = 0$.

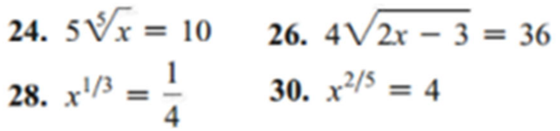
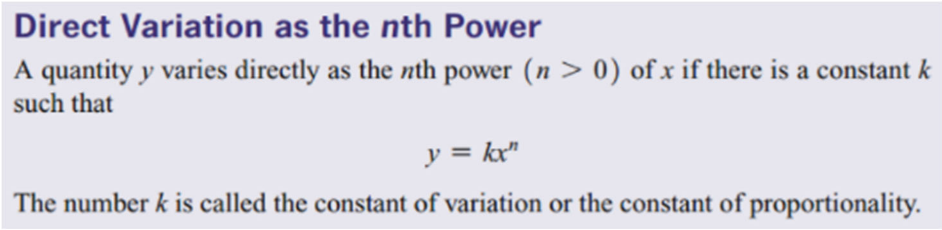
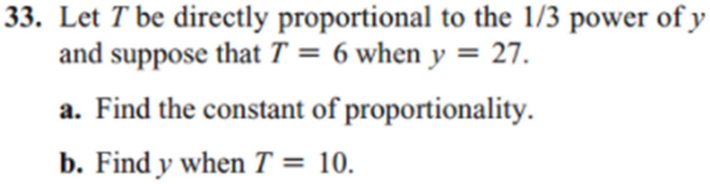
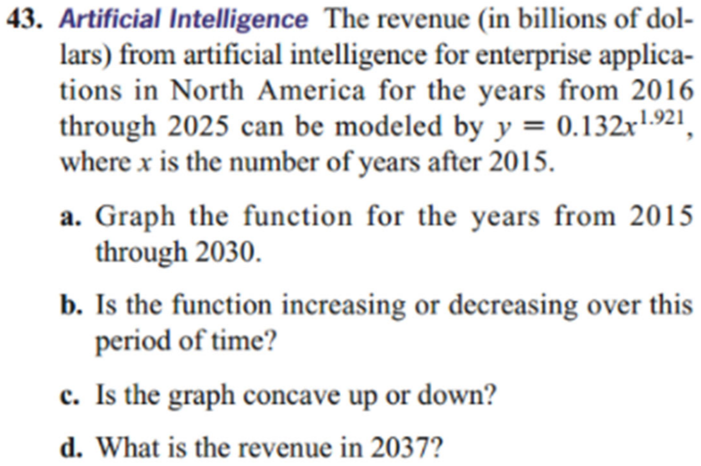
59. $f(x) = 3x^2 - 5x - 2$ **62.** $f(x) = 4x^2 + 25$

Solve applications involving quadratic equations.	 *In Exercises 63–74, solve analytically and then check graphically.* **63.** **Flight of a Ball** If a ball is thrown upward at 96 feet per second from the top of a building that is 100 feet high, the height of the ball can be modeled by $S(t) = 100 + 96t - 16t^2$ feet, where t is the number of seconds after the ball is thrown. How long after the ball is thrown is the height 228 feet? **69.** **Wind and Pollution** The amount of particulate pollution p from a power plant in the air above the plant depends on the wind speed s, among other things, with the relationship between p and s approximated by $p = 25 - 0.01s^2$, with s in miles per hour. **a.** Find the value(s) of s that will make $p = 0$. **b.** What does $p = 0$ mean in this application? **c.** What solution to $0 = 25 - 0.01s^2$ makes sense in the context of this application?

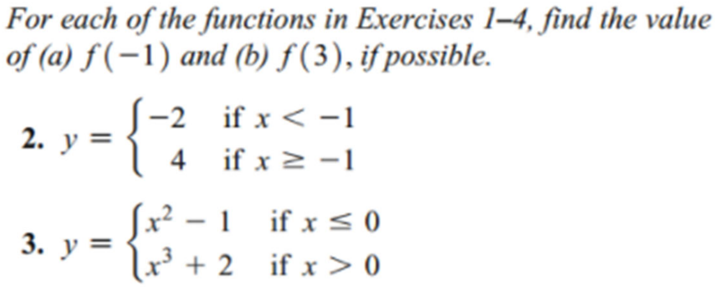
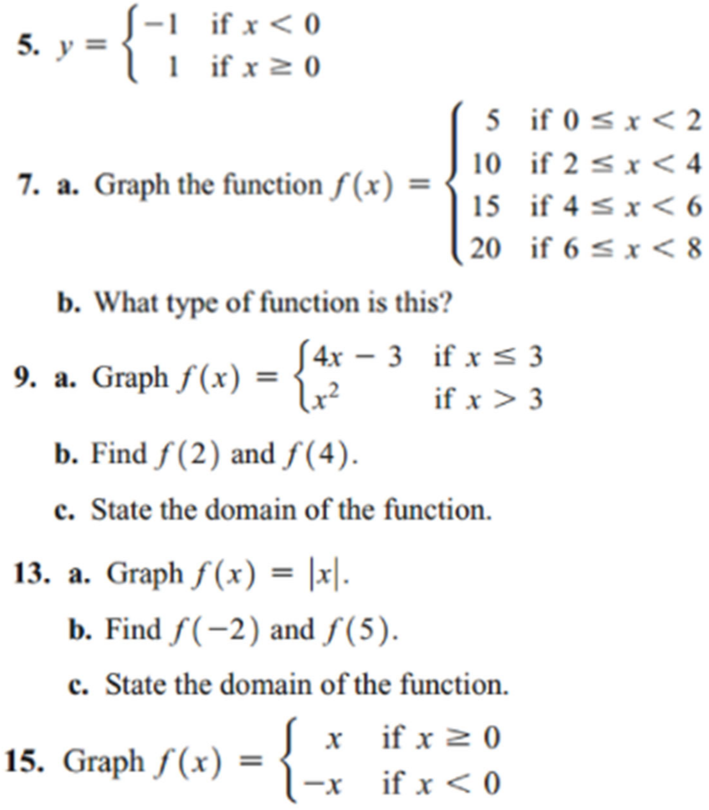
3.3 Power and Root Functions

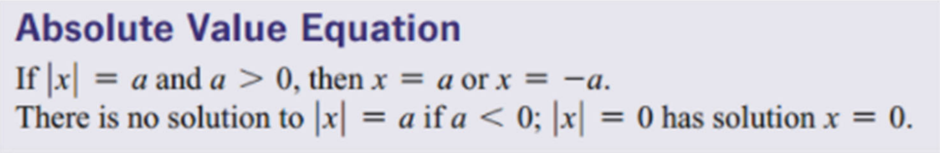
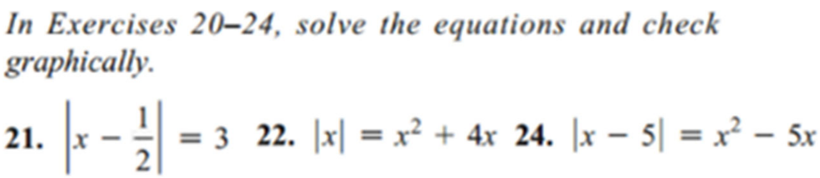
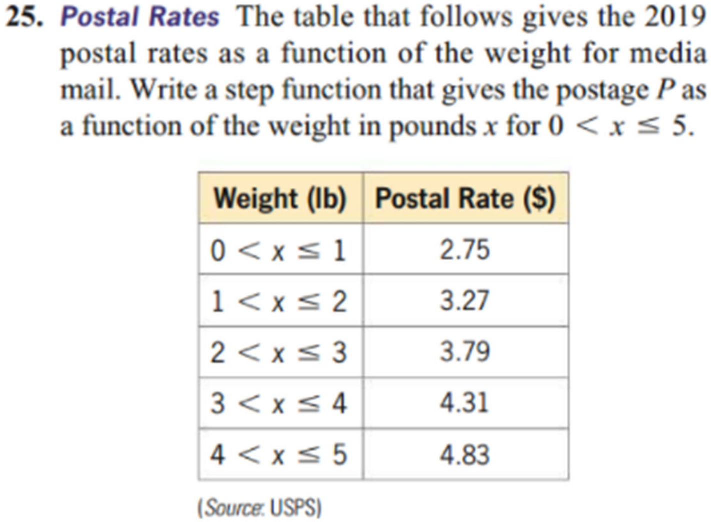
Learning Outcomes and Notes	**Examples**
Identify power functions and describe characteristics of power functions.  **Power Functions** A **power function** is a function of the form $y = ax^b$, where a and b are real numbers, $b \neq 0$.	 **1.** Which of the following are power functions? **a.** $y = 3x^4$ **b.** $d = q^4$ **c.** $f = 3^x$ **d.** $g(x) = x^{1/4}$ **2.** Rewrite each of the following as power functions of the form $y = ax^b$. **c.** $y = 3\sqrt[4]{x^5}$ **d.** $y = \frac{1}{x^2}$

<table>
<tr><td></td><td>
3. Determine whether the function $y = 4x^3$ is increasing or decreasing for
a. $x < 0$. **b.** $x > 0$.
4. Determine whether the function $y = -3x^4$ is increasing or decreasing for
a. $x < 0$. **b.** $x > 0$.
5. Complete the table of values for the function $y = x^{3.5}$, if possible. Round to one decimal place.
<table><tr><td>x</td><td>− 1</td><td>0</td><td>1</td><td>2</td><td>10</td></tr><tr><td>$y = x^{3.5}$</td><td></td><td></td><td></td><td></td><td></td></tr></table>
7. For which value(s) of x in Exercise 5 is the function undefined? Explain.
For each of the functions in Exercises 9–12, determine if the function is concave up or concave down in the first quadrant.
10. $y = x^{3/2}$ **12.** $y = x^{0.6}$
13. Give an example of a power function that is increasing and concave down.
14. Give an example of a power function that is decreasing and concave up.</td></tr>
<tr><td>Graph power functions.</td><td>
16. Graph the function $y = x^{-1.5}$. What are the domain and range? Does this graph have any asymptotes? If so, where?
18. Graph the function $f(x) = -3\sqrt[4]{x}$. What are the domain and range?</td></tr>
<tr><td>Solve equations involving integer exponents.</td><td>
In Exercises 19–30, solve the equation and check for extraneous solutions.
20. $\frac{1}{8}x^{-5} = 384$ **21.** $3(3x - 2)^4 = 768$ **22.** $x^{2/3} - \frac{3}{4} = -\frac{1}{2}$</td></tr>
</table>

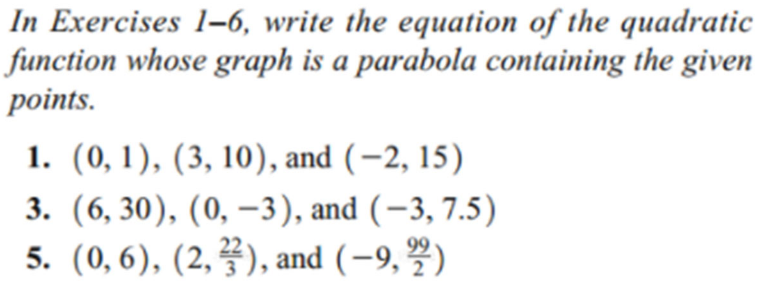
Solve equations involving rational exponents.	 **24.** $5\sqrt[5]{x} = 10$ **26.** $4\sqrt{2x - 3} = 36$ **28.** $x^{1/3} = \frac{1}{4}$ **30.** $x^{2/5} = 4$
Solve direct variation problems involving power functions.  **Direct Variation as the *n*th Power** A quantity y varies directly as the nth power $(n > 0)$ of x if there is a constant k such that $y = kx^n$ The number k is called the constant of variation or the constant of proportionality.	 **33.** Let T be directly proportional to the 1/3 power of y and suppose that $T = 6$ when $y = 27$. **a.** Find the constant of proportionality. **b.** Find y when $T = 10$.
Solve applications involving power functions.	 **43.** ***Artificial Intelligence*** The revenue (in billions of dollars) from artificial intelligence for enterprise applications in North America for the years from 2016 through 2025 can be modeled by $y = 0.132x^{1.921}$, where x is the number of years after 2015. **a.** Graph the function for the years from 2015 through 2030. **b.** Is the function increasing or decreasing over this period of time? **c.** Is the graph concave up or down? **d.** What is the revenue in 2037?

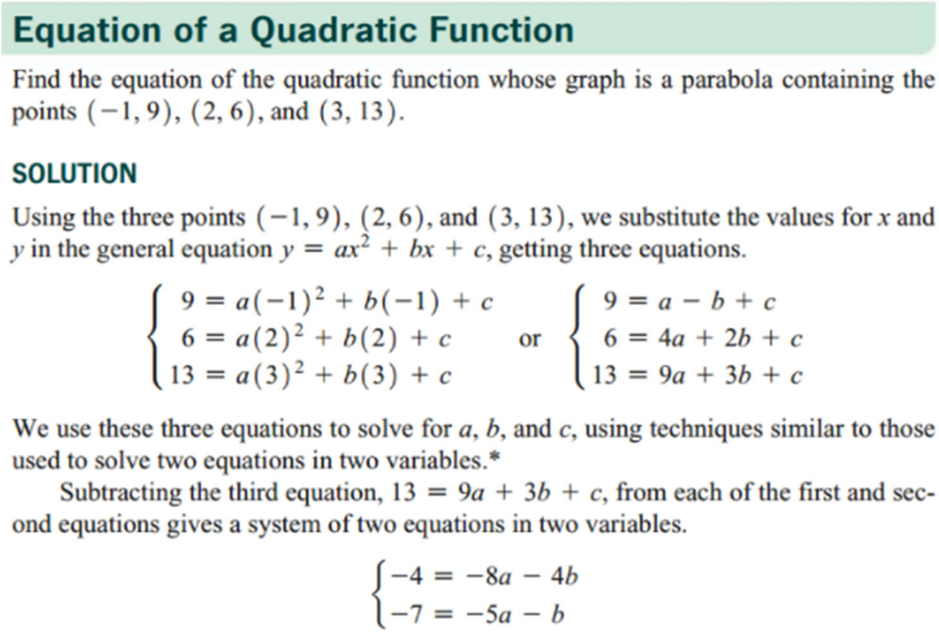
3.4 Piecewise-Defined Functions and Absolute Value Functions

Learning Outcomes and Notes	**Examples**
Evaluate piecewise-defined functions.	 *For each of the functions in Exercises 1–4, find the value of (a) $f(-1)$ and (b) $f(3)$, if possible.* **2.** $y = \begin{cases} -2 & \text{if } x < -1 \\ 4 & \text{if } x \geq -1 \end{cases}$ **3.** $y = \begin{cases} x^2 - 1 & \text{if } x \leq 0 \\ x^3 + 2 & \text{if } x > 0 \end{cases}$
Graph piecewise-defined functions.	 **5.** $y = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ **7. a.** Graph the function $f(x) = \begin{cases} 5 & \text{if } 0 \leq x < 2 \\ 10 & \text{if } 2 \leq x < 4 \\ 15 & \text{if } 4 \leq x < 6 \\ 20 & \text{if } 6 \leq x < 8 \end{cases}$ **b.** What type of function is this? **9. a.** Graph $f(x) = \begin{cases} 4x - 3 & \text{if } x \leq 3 \\ x^2 & \text{if } x > 3 \end{cases}$ **b.** Find $f(2)$ and $f(4)$. **c.** State the domain of the function. **13. a.** Graph $f(x) = \lvert x \rvert$. **b.** Find $f(-2)$ and $f(5)$. **c.** State the domain of the function. **15.** Graph $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

<table>
<tr>
<td>Solve absolute value equations.

Absolute Value Equation
If $|x| = a$ and $a > 0$, then $x = a$ or $x = -a$.
There is no solution to $|x| = a$ if $a < 0$; $|x| = 0$ has solution $x = 0$.</td>
<td>
<i>In Exercises 20–24, solve the equations and check graphically.</i>
21. $\left|x - \frac{1}{2}\right| = 3$ 22. $|x| = x^2 + 4x$ 24. $|x - 5| = x^2 - 5x$</td>
</tr>
<tr>
<td>Solve applications involving piecewise-defined functions.</td>
<td>
25. Postal Rates The table that follows gives the 2019 postal rates as a function of the weight for media mail. Write a step function that gives the postage P as a function of the weight in pounds x for $0 < x \le 5$.
<table>
<tr><th>Weight (lb)</th><th>Postal Rate ($)</th></tr>
<tr><td>$0 < x \le 1$</td><td>2.75</td></tr>
<tr><td>$1 < x \le 2$</td><td>3.27</td></tr>
<tr><td>$2 < x \le 3$</td><td>3.79</td></tr>
<tr><td>$3 < x \le 4$</td><td>4.31</td></tr>
<tr><td>$4 < x \le 5$</td><td>4.83</td></tr>
</table>
(<i>Source:</i> USPS)</td>
</tr>
</table>

3.5 Quadratic and Power Models

<table>
<tr><th>Learning Outcomes and Notes</th><th>Examples</th></tr>
<tr>
<td>Find the quadratic function whose graph contains specified points.</td>
<td>
<i>In Exercises 1–6, write the equation of the quadratic function whose graph is a parabola containing the given points.</i>
1. $(0, 1)$, $(3, 10)$, and $(-2, 15)$
3. $(6, 30)$, $(0, -3)$, and $(-3, 7.5)$
5. $(0, 6)$, $\left(2, \frac{22}{3}\right)$, and $\left(-9, \frac{99}{2}\right)$</td>
</tr>
</table>



Equation of a Quadratic Function

Find the equation of the quadratic function whose graph is a parabola containing the points $(-1, 9)$, $(2, 6)$, and $(3, 13)$.

SOLUTION

Using the three points $(-1, 9)$, $(2, 6)$, and $(3, 13)$, we substitute the values for x and y in the general equation $y = ax^2 + bx + c$, getting three equations.

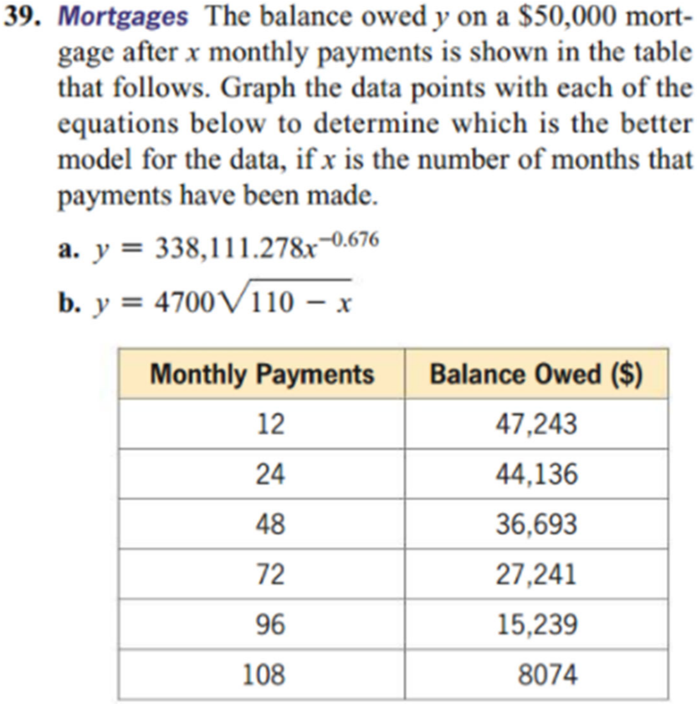
$$\begin{cases} 9 = a(-1)^2 + b(-1) + c \\ 6 = a(2)^2 + b(2) + c \\ 13 = a(3)^2 + b(3) + c \end{cases} \quad \text{or} \quad \begin{cases} 9 = a - b + c \\ 6 = 4a + 2b + c \\ 13 = 9a + 3b + c \end{cases}$$

We use these three equations to solve for a, b, and c, using techniques similar to those used to solve two equations in two variables.*

Subtracting the third equation, $13 = 9a + 3b + c$, from each of the first and second equations gives a system of two equations in two variables.

$$\begin{cases} -4 = -8a - 4b \\ -7 = -5a - b \end{cases}$$

Solve applications involving quadratic and power models.

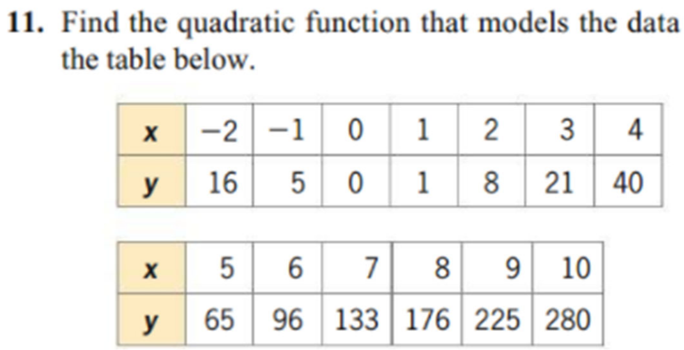


39. **Mortgages** The balance owed y on a \$50,000 mortgage after x monthly payments is shown in the table that follows. Graph the data points with each of the equations below to determine which is the better model for the data, if x is the number of months that payments have been made.

a. $y = 338{,}111.278x^{-0.676}$

b. $y = 4700\sqrt{110 - x}$

Monthly Payments	Balance Owed (\$)
12	47,243
24	44,136
48	36,693
72	27,241
96	15,239
108	8074

<table>
<tr>
<td>Identify quadratic functions given a table of function values.</td>
<td>


11. Find the quadratic function that models the data in the table below.

<table>
<tr><td>x</td><td>−2</td><td>−1</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>y</td><td>16</td><td>5</td><td>0</td><td>1</td><td>8</td><td>21</td><td>40</td></tr>
</table>

<table>
<tr><td>x</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td></tr>
<tr><td>y</td><td>65</td><td>96</td><td>133</td><td>176</td><td>225</td><td>280</td></tr>
</table>
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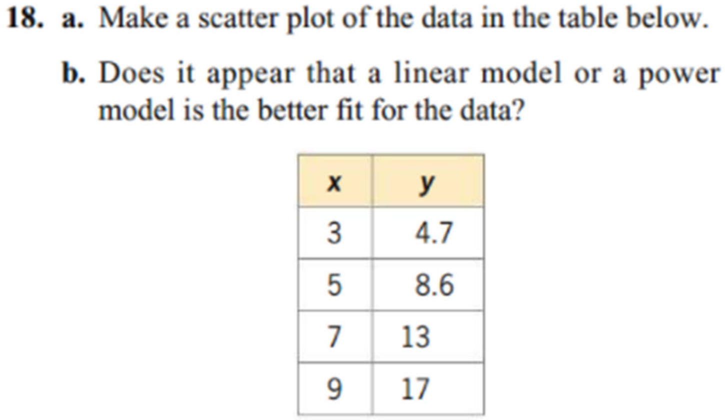
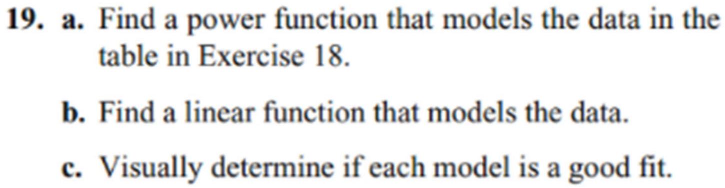
</tr>
<tr>
<td>Find linear, quadratic, or power functions that model data in a table.</td>
<td>


18. a. Make a scatter plot of the data in the table below.

b. Does it appear that a linear model or a power model is the better fit for the data?

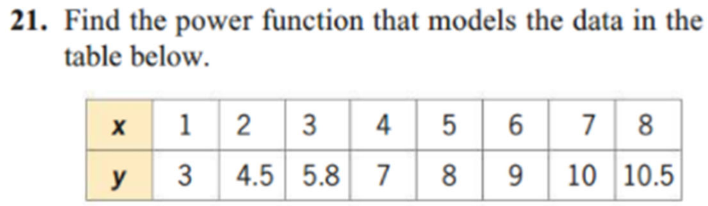
<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>3</td><td>4.7</td></tr>
<tr><td>5</td><td>8.6</td></tr>
<tr><td>7</td><td>13</td></tr>
<tr><td>9</td><td>17</td></tr>
</table>



19. a. Find a power function that models the data in the table in Exercise 18.

b. Find a linear function that models the data.

c. Visually determine if each model is a good fit.



21. Find the power function that models the data in the table below.

<table>
<tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td></tr>
<tr><td>y</td><td>3</td><td>4.5</td><td>5.8</td><td>7</td><td>8</td><td>9</td><td>10</td><td>10.5</td></tr>
</table>
</td>
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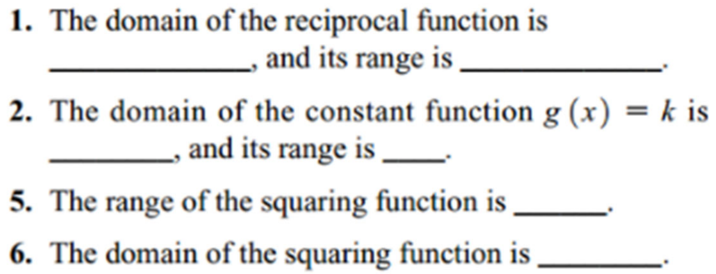
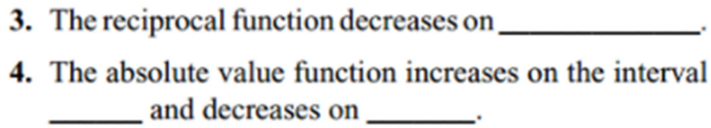
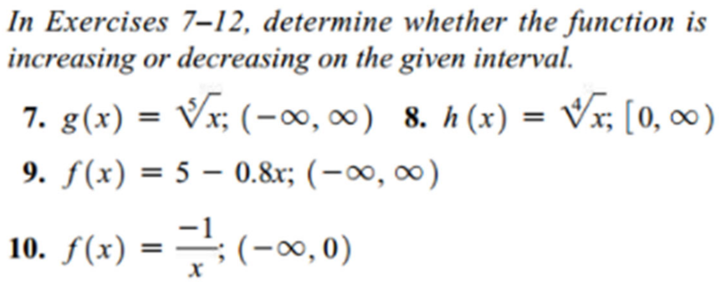
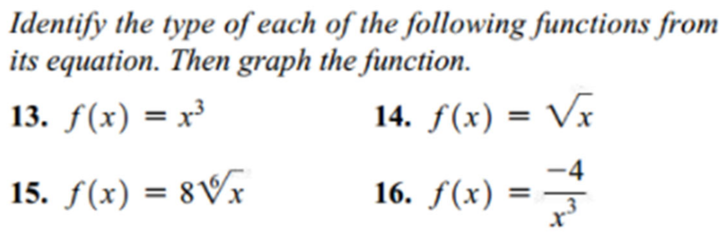
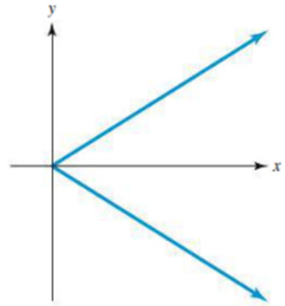
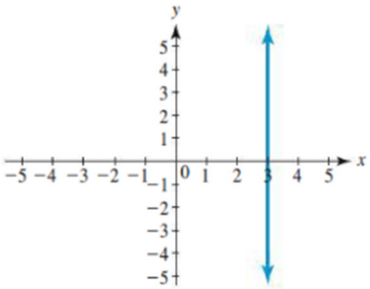
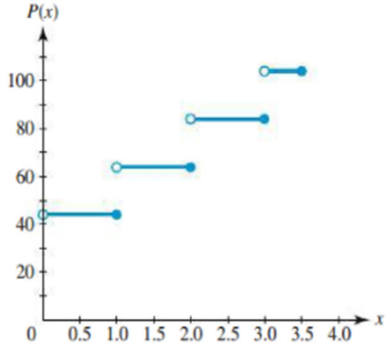
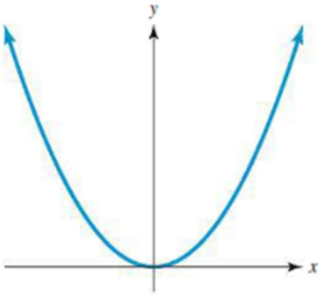
Chapter 4: Additional Topics with Functions

Chapter 4 Algebra Toolbox

Learning Outcomes and Notes	**Examples**
Identify the domain and range of a given type of function.	 **1.** The domain of the reciprocal function is ______________, and its range is ______________. **2.** The domain of the constant function $g(x) = k$ is _________, and its range is _____. **5.** The range of the squaring function is ______. **6.** The domain of the squaring function is _________.
Determine over what intervals a function is increasing or decreasing.	 **3.** The reciprocal function decreases on ______________. **4.** The absolute value function increases on the interval ______ and decreases on _______.
Determine if a function is increasing or decreasing over given intervals.	 *In Exercises 7–12, determine whether the function is increasing or decreasing on the given interval.* **7.** $g(x) = \sqrt[5]{x}; (-\infty, \infty)$ **8.** $h(x) = \sqrt[4]{x}; [0, \infty)$ **9.** $f(x) = 5 - 0.8x; (-\infty, \infty)$ **10.** $f(x) = \frac{-1}{x}; (-\infty, 0)$
Identify a function from its equation and graph it.	 *Identify the type of each of the following functions from its equation. Then graph the function.* **13.** $f(x) = x^3$ **14.** $f(x) = \sqrt{x}$ **15.** $f(x) = 8\sqrt[6]{x}$ **16.** $f(x) = \frac{-4}{x^3}$

Determine if a given graph represents a function. *Which of the following graphs represents a function?* **17.**  **18.**  	*Which of the following graphs represents a function?* **21.**  **22.** 

4.4 Additional Equations and Inequalities

Learning Outcomes and Notes	**Examples**
Solve radical equations. **Solving Radical Equations** 1. Isolate a single radical on one side of the equation. 2. Square both sides of the equation. (Or raise both sides to a power that is equal to the index of the radical.) 3. If a radical remains, repeat steps 1 and 2. 4. Solve the resulting equation. 5. All solutions must be checked in the original equation, and only those that satisfy the original equation are actual solutions.	*In Exercises 1–20, solve the equations algebraically and check graphically or by substitution.* **2.** $\sqrt{3x^2 + 4} - 2x = 0$ **3.** $\sqrt[3]{x - 1} = -2$ **5.** $\sqrt{3x - 2} + 2 = x$ **7.** $\sqrt[3]{4x + 5} = \sqrt[3]{x^2 - 7}$ **10.** $\sqrt{x} - 10 = -\sqrt{x - 20}$
Solve equations with rational powers.	**12.** $(x - 5)^{3/2} = 64$ **14.** $6x^{5/3} - 18 = 0$

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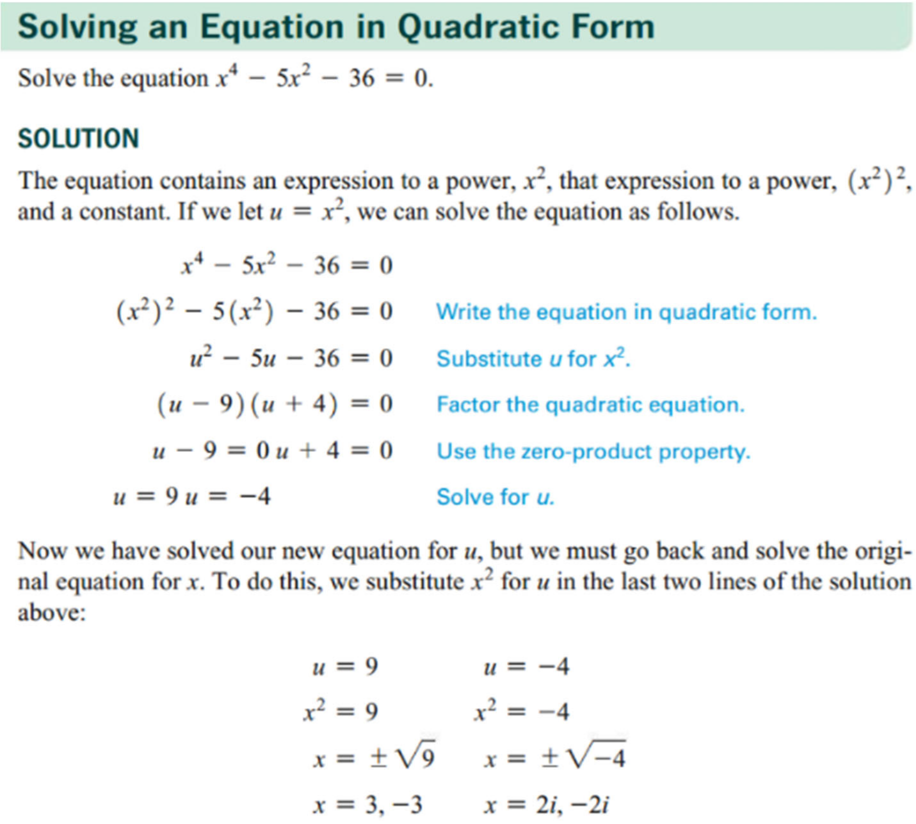
Solve equations that are in quadratic form.



Solving an Equation in Quadratic Form

Solve the equation $x^4 - 5x^2 - 36 = 0$.

SOLUTION

The equation contains an expression to a power, x^2, that expression to a power, $(x^2)^2$, and a constant. If we let $u = x^2$, we can solve the equation as follows.

$$\begin{aligned} x^4 - 5x^2 - 36 &= 0 \\ (x^2)^2 - 5(x^2) - 36 &= 0 && \text{Write the equation in quadratic form.} \\ u^2 - 5u - 36 &= 0 && \text{Substitute } u \text{ for } x^2. \\ (u - 9)(u + 4) &= 0 && \text{Factor the quadratic equation.} \\ u - 9 = 0 \; u + 4 &= 0 && \text{Use the zero-product property.} \\ u = 9 \; u &= -4 && \text{Solve for } u. \end{aligned}$$

Now we have solved our new equation for u, but we must go back and solve the original equation for x. To do this, we substitute x^2 for u in the last two lines of the solution above:

$$\begin{aligned} u &= 9 & u &= -4 \\ x^2 &= 9 & x^2 &= -4 \\ x &= \pm\sqrt{9} & x &= \pm\sqrt{-4} \\ x &= 3, -3 & x &= 2i, -2i \end{aligned}$$

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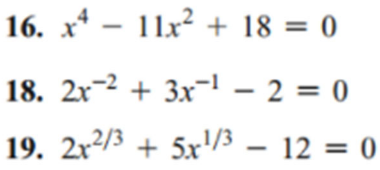

16. $x^4 - 11x^2 + 18 = 0$

18. $2x^{-2} + 3x^{-1} - 2 = 0$

19. $2x^{2/3} + 5x^{1/3} - 12 = 0$

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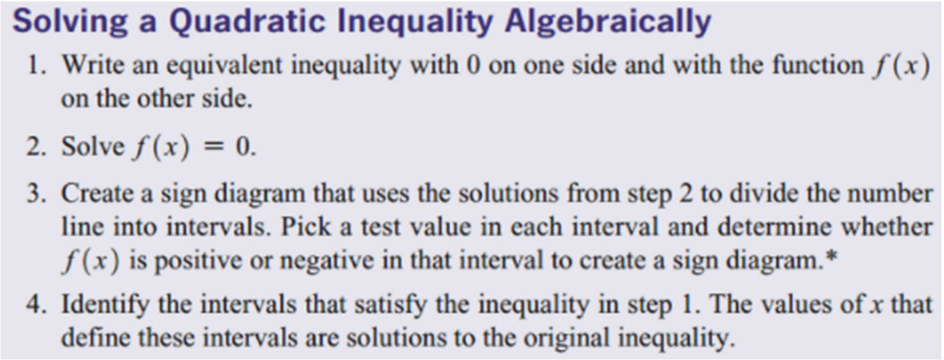
Solve quadratic inequalities.



Solving a Quadratic Inequality Algebraically

1. Write an equivalent inequality with 0 on one side and with the function $f(x)$ on the other side.
2. Solve $f(x) = 0$.
3. Create a sign diagram that uses the solutions from step 2 to divide the number line into intervals. Pick a test value in each interval and determine whether $f(x)$ is positive or negative in that interval to create a sign diagram.*
4. Identify the intervals that satisfy the inequality in step 1. The values of x that define these intervals are solutions to the original inequality.

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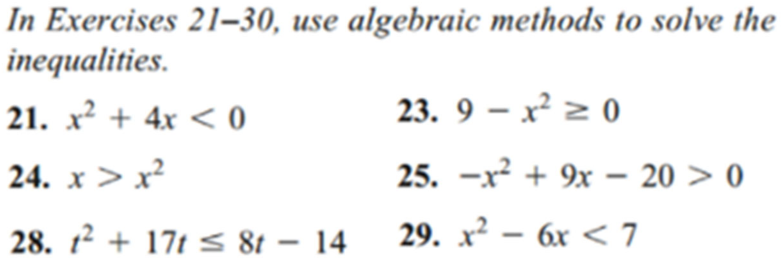

In Exercises 21–30, use algebraic methods to solve the inequalities.

21. $x^2 + 4x < 0$ **23.** $9 - x^2 \geq 0$

24. $x > x^2$ **25.** $-x^2 + 9x - 20 > 0$

28. $t^2 + 17t \leq 8t - 14$ **29.** $x^2 - 6x < 7$

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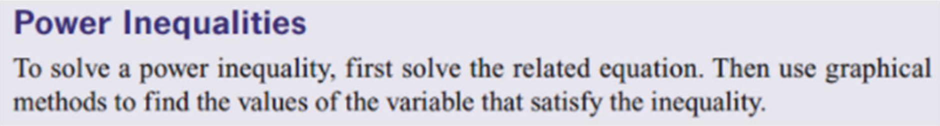
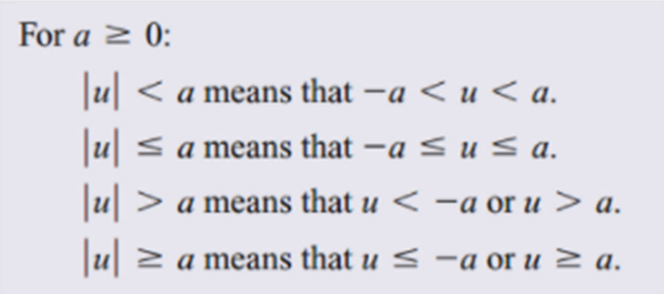
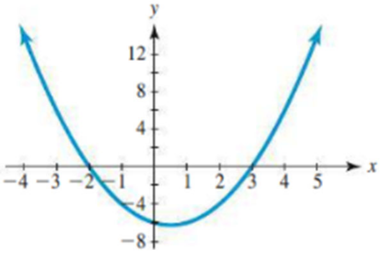
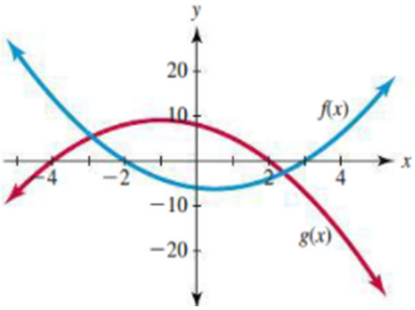
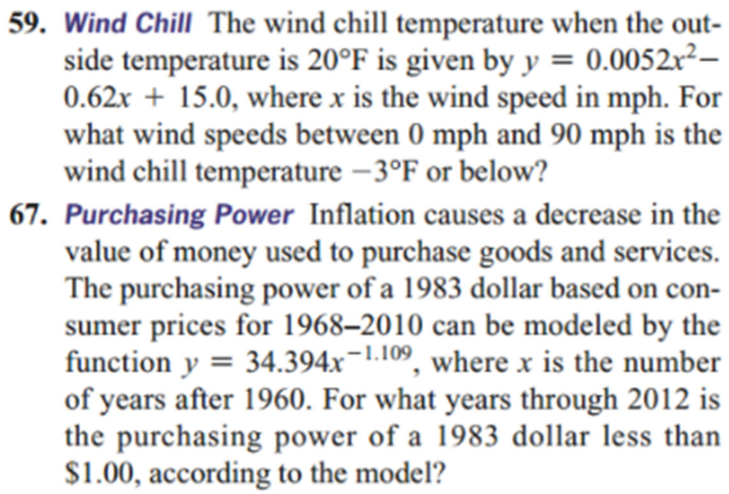
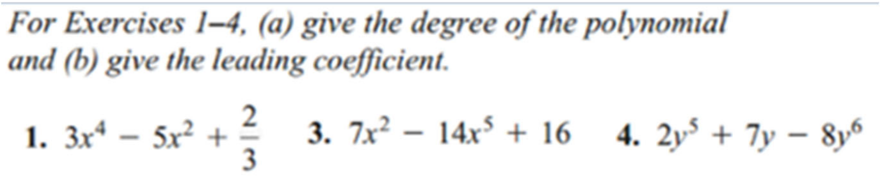
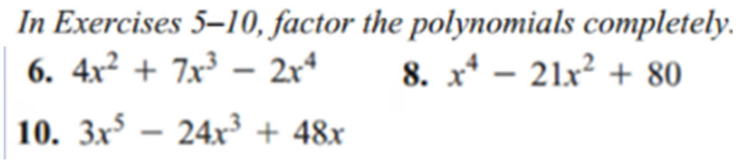
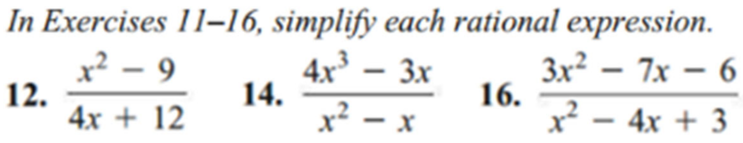
</table>

Solve power inequalities.  **Power Inequalities** To solve a power inequality, first solve the related equation. Then use graphical methods to find the values of the variable that satisfy the inequality.	*In Exercises 35–42, solve the inequalities by using algebraic and graphical methods.* **36.** $(x-2)^3 \ge -2$ **38.** $(x+5)^4 > 16$
Solve absolute value inequalities.  For $a \ge 0$: $\lvert u\rvert < a$ means that $-a < u < a$. $\lvert u\rvert \le a$ means that $-a \le u \le a$. $\lvert u\rvert > a$ means that $u < -a$ or $u > a$. $\lvert u\rvert \ge a$ means that $u \le -a$ or $u \ge a$.	**39.** $\lvert 2x-1\rvert < 3$ **40.** $\lvert 3x+1\rvert \le 5$
Solve applications involving additional equations and inequalities.	*In Exercises 43–46, you are given the graphs of several functions of the form $f(x) = ax^2 + bx + c$ for different values of a, b, and c. For each function, (a) solve $f(x) \ge 0$, and (b) solve $f(x) < 0$.* **43.**  *In Exercises 47 and 48, you are given the graphs of two functions $f(x)$ and $g(x)$. Solve $f(x) \le g(x)$.* **47.** 

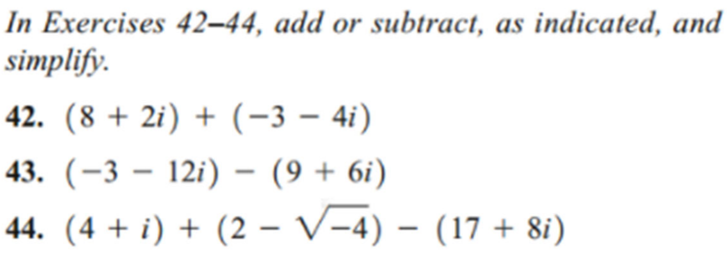
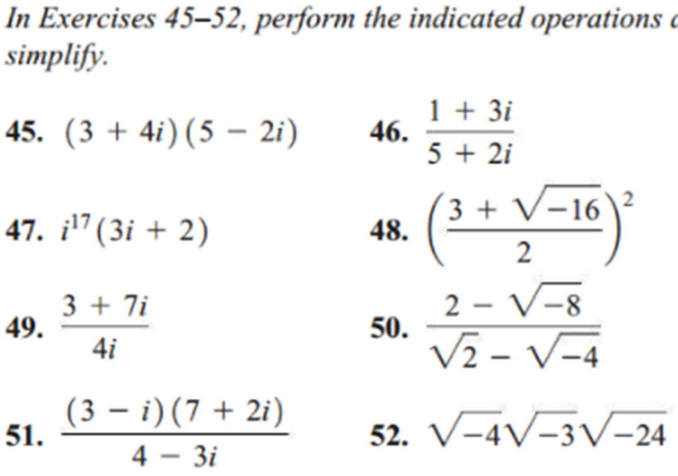
Answer interactive figure questions involving polynomial inequalities.	 **59.** **Wind Chill** The wind chill temperature when the outside temperature is 20°F is given by $y = 0.0052x^2 - 0.62x + 15.0$, where x is the wind speed in mph. For what wind speeds between 0 mph and 90 mph is the wind chill temperature −3°F or below? **67.** **Purchasing Power** Inflation causes a decrease in the value of money used to purchase goods and services. The purchasing power of a 1983 dollar based on consumer prices for 1968–2010 can be modeled by the function $y = 34.394x^{-1.109}$, where x is the number of years after 1960. For what years through 2012 is the purchasing power of a 1983 dollar less than \$1.00, according to the model?

Chapter 6: Higher-Degree Polynomial and Rational Functions

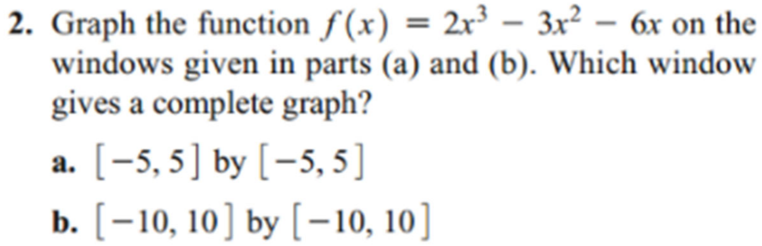
Chapter 6 Algebra Toolbox

Learning Outcomes and Notes	**Examples**
Identify the degrees and leading coefficients of polynomials.	 *For Exercises 1–4, (a) give the degree of the polynomial and (b) give the leading coefficient.* **1.** $3x^4 - 5x^2 + \frac{2}{3}$ **3.** $7x^2 - 14x^5 + 16$ **4.** $2y^5 + 7y - 8y^6$
Factor higher-degree polynomials completely.	 *In Exercises 5–10, factor the polynomials completely.* **6.** $4x^2 + 7x^3 - 2x^4$ **8.** $x^4 - 21x^2 + 80$ **10.** $3x^5 - 24x^3 + 48x$
Simplify rational expressions.	 *In Exercises 11–16, simplify each rational expression.* **12.** $\frac{x^2 - 9}{4x + 12}$ **14.** $\frac{4x^3 - 3x}{x^2 - x}$ **16.** $\frac{3x^2 - 7x - 6}{x^2 - 4x + 3}$

Multiply and divide rational expressions.	**17.** $\frac{6x^3}{8y^3} \cdot \frac{16x}{9y^2} \cdot \frac{15y^4}{x^3}$ **18.** $\frac{x-3}{x^3} \cdot \frac{x^2-4x}{x^2-7x+12}$ **19.** $(x^2-4) \cdot \frac{2x-3}{x+2}$ **20.** $(x^2-x-6) \div \frac{9-x^2}{x^2+3x}$ **22.** $\frac{6x^2}{4x^2y-12xy} \div \frac{3x^2+12x}{x^2+x-12}$ **23.** $\frac{x^2+x}{x^2-5x+6} \cdot \frac{x^2-2x-3}{2x+4} \div \frac{x^3-3x^2}{4-x^2}$
Add and subtract rational expressions.	**25.** $\frac{2x+3}{x^2-1} + \frac{4x+3}{x^2-1}$ **29.** $\frac{5x}{x^4-16} + \frac{8x}{x+2}$ **32.** $\frac{x-7}{x^2-9x+20} + \frac{x+2}{x^2-5x+4}$ **30.** $\frac{x-1}{x+1} - \frac{2}{x^2+x}$ **33.** $\frac{2x+1}{4x-2} + \frac{5}{2x} - \frac{x+1}{2x^2-x}$
Simplify complex fractions.	*In Exercises 34–37, simplify the complex fraction.* **35.** $\dfrac{\frac{5}{2y}+\frac{3}{y}}{\frac{1}{4}+\frac{1}{3y}}$ **37.** $\dfrac{1-\frac{2}{x-2}}{x-6+\frac{10}{x+1}}$
Divide polynomials using long division.	*In Exercises 38–41, perform the long division.* **38.** $(x^5+x^3-1) \div (x+1)$ **40.** $(3x^5-x^4+5x-1) \div (x^2-2)$

Add and subtract complex numbers.	 *In Exercises 42–44, add or subtract, as indicated, and simplify.* **42.** $(8 + 2i) + (-3 - 4i)$ **43.** $(-3 - 12i) - (9 + 6i)$ **44.** $(4 + i) + (2 - \sqrt{-4}) - (17 + 8i)$
Multiply and divide complex numbers.	 *In Exercises 45–52, perform the indicated operations and simplify.* **45.** $(3 + 4i)(5 - 2i)$ **46.** $\frac{1 + 3i}{5 + 2i}$ **47.** $i^{17}(3i + 2)$ **48.** $\left(\frac{3 + \sqrt{-16}}{2}\right)^2$ **49.** $\frac{3 + 7i}{4i}$ **50.** $\frac{2 - \sqrt{-8}}{\sqrt{2} - \sqrt{-4}}$ **51.** $\frac{(3 - i)(7 + 2i)}{4 - 3i}$ **52.** $\sqrt{-4}\sqrt{-3}\sqrt{-24}$

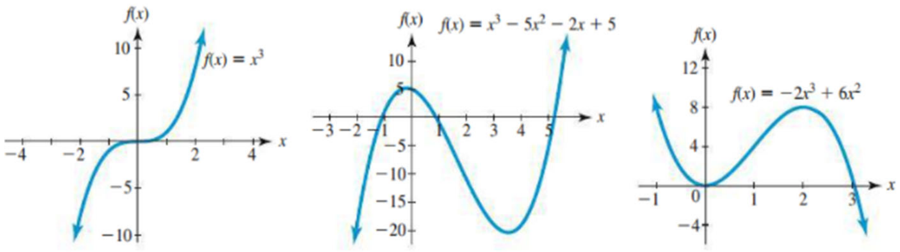
6.1 Higher-Degree Polynomial Functions

Learning Outcomes and Notes	**Examples**
Graph cubic functions.	 **2.** Graph the function $f(x) = 2x^3 - 3x^2 - 6x$ on the windows given in parts (a) and (b). Which window gives a complete graph? **a.** $[-5, 5]$ by $[-5, 5]$ **b.** $[-10, 10]$ by $[-10, 10]$

A **cubic function** in the variable x has the form

$$f(x) = ax^3 + bx^2 + cx + d \qquad (a \neq 0)$$

The basic cubic function $f(x) = x^3$ was discussed in Section 3.3. The graph of this function, shown in Figure 6.3(a), is one of the possible shapes of the graph of a cubic function. The graphs of two other cubic functions are shown in Figures 6.3(b) and (c).



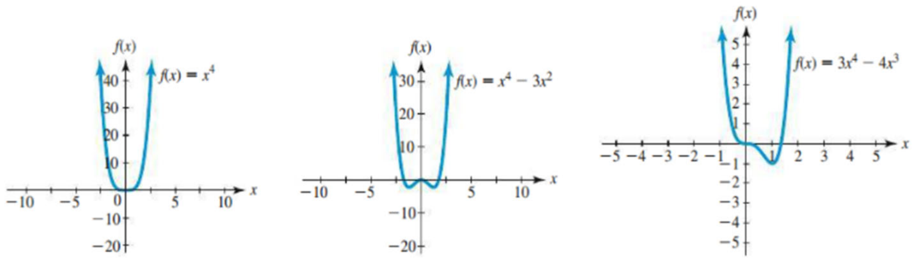
(a) (b) (c)

Figure 6.3

Graph quartic functions.

Quartic Functions

A polynomial function of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e, a \neq 0$, is a fourth-degree, or **quartic**, function. The basic quartic function $f(x) = x^4$ has the graph shown in Figure 6.7(a). Notice that this graph resembles a parabola, although it is not actually a parabola (recall the shape of a parabola, which was discussed in Section 3.1). The graph in Figure 6.7(a) is one of the possible shapes of the graph of a quartic function. The graphs of two additional quartic functions are shown in Figures 6.7(b) and (c).



(a) (b) (c)

Figure 6.7

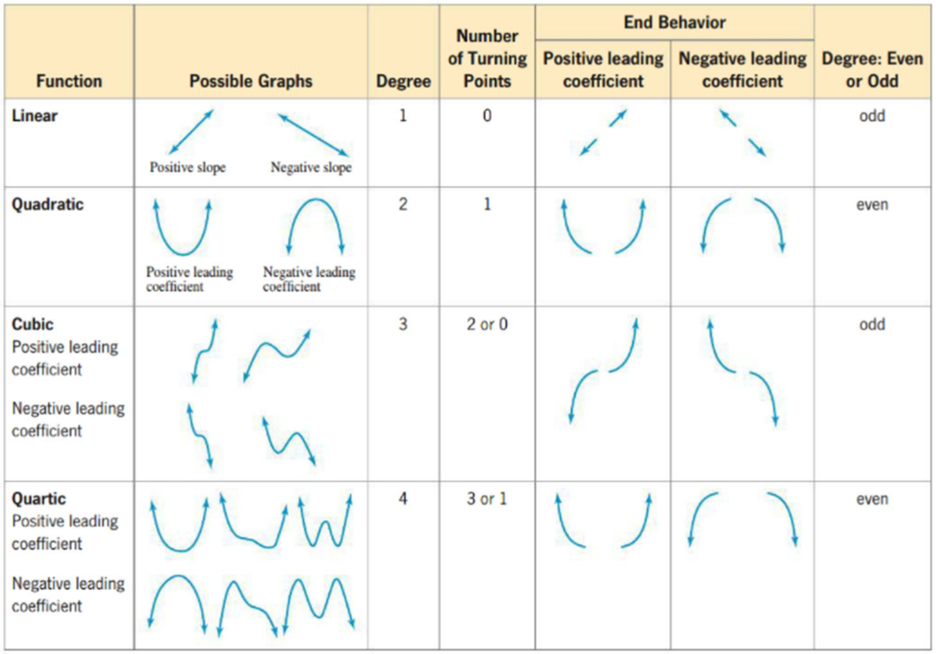
3. Graph the function $g(x) = 3x^4 - 12x^2$ on the windows given in parts (a) and (b). Which window gives a complete graph?

a. $[-5, 5]$ by $[-5, 5]$ **b.** $[-3, 3]$ by $[-12, 10]$

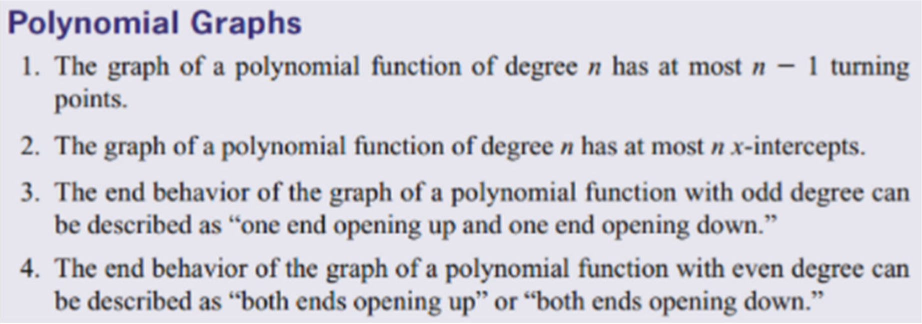
Identify the degree, coefficient, and intercepts of graphs of higher-degree polynomial functions.

In general, the graph of a polynomial function of degree n has at most n x-intercepts.

Table 6.3

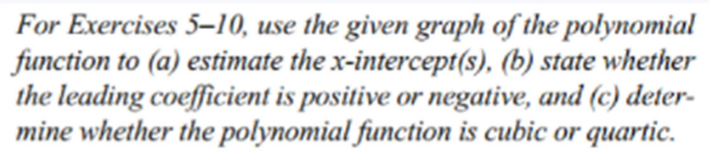


Function	Possible Graphs	Degree	Number of Turning Points	End Behavior		Degree: Even or Odd
				Positive leading coefficient	Negative leading coefficient	
Linear	Positive slope Negative slope	1	0			odd
Quadratic	Positive leading coefficient Negative leading coefficient	2	1			even
Cubic Positive leading coefficient Negative leading coefficient		3	2 or 0			odd
Quartic Positive leading coefficient Negative leading coefficient		4	3 or 1			even



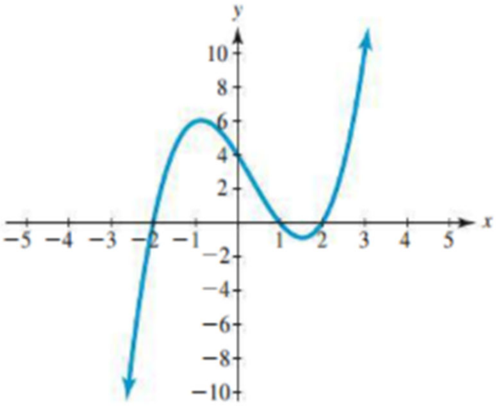
Polynomial Graphs

1. The graph of a polynomial function of degree n has at most $n - 1$ turning points.
2. The graph of a polynomial function of degree n has at most n x-intercepts.
3. The end behavior of the graph of a polynomial function with odd degree can be described as "one end opening up and one end opening down."
4. The end behavior of the graph of a polynomial function with even degree can be described as "both ends opening up" or "both ends opening down."



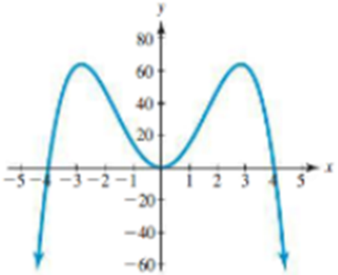
For Exercises 5–10, use the given graph of the polynomial function to (a) estimate the x-intercept(s), (b) state whether the leading coefficient is positive or negative, and (c) determine whether the polynomial function is cubic or quartic.

5.

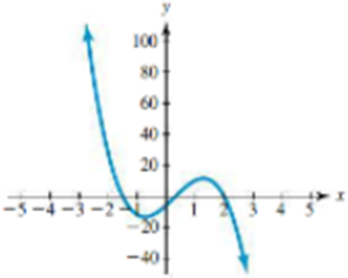


Identify the graphs of higher-degree polynomial functions.

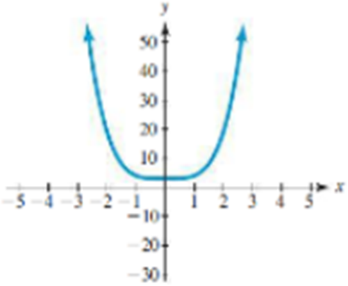
D.



E.



F.



For Exercises 11–16, match the polynomial function with its graph (below or on the next page).

11. $y = 2x^3 + 3x^2 - 23x - 12$

12. $y = -2x^3 - 8x^2 + 0.5x + 2$

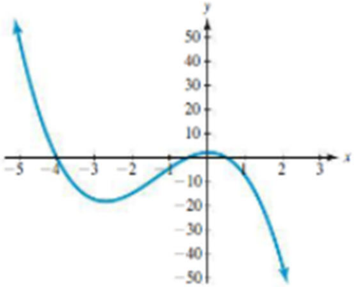
13. $y = -6x^3 + 5x^2 + 17x - 6$

14. $y = x^4 - 0.5x^3 - 14.5x^2 + 17x + 12$

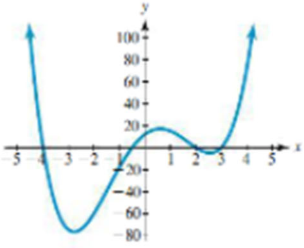
15. $y = x^4 + 3$

16. $y = -x^4 + 16x^2$

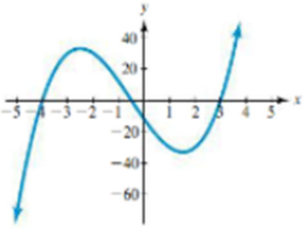
A.

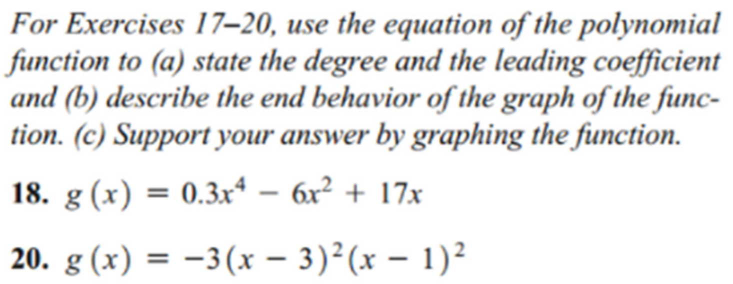


B.



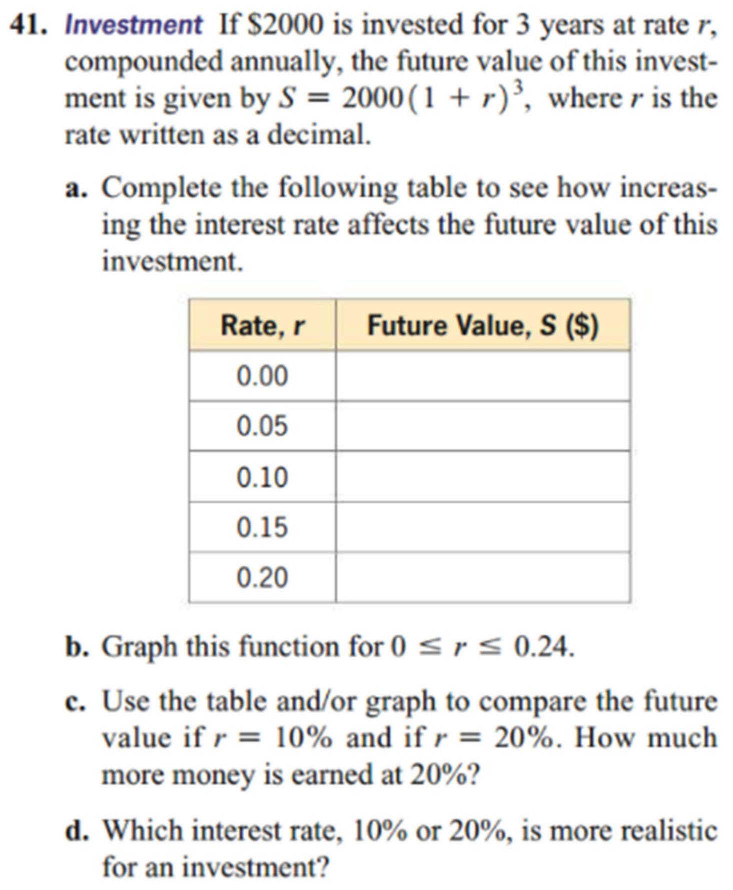
C.



<table>
<tr>
<td>Identify the degree, coefficient, and intercepts of equations of higher-degree polynomial functions.</td>
<td>


For Exercises 17–20, use the equation of the polynomial function to (a) state the degree and the leading coefficient and (b) describe the end behavior of the graph of the function. (c) Support your answer by graphing the function.

18. $g(x) = 0.3x^4 - 6x^2 + 17x$

20. $g(x) = -3(x - 3)^2(x - 1)^2$
</td>
</tr>
<tr>
<td>Solve applications involving higher-degree polynomial functions.</td>
<td>


41. **Investment** If \$2000 is invested for 3 years at rate r, compounded annually, the future value of this investment is given by $S = 2000(1 + r)^3$, where r is the rate written as a decimal.

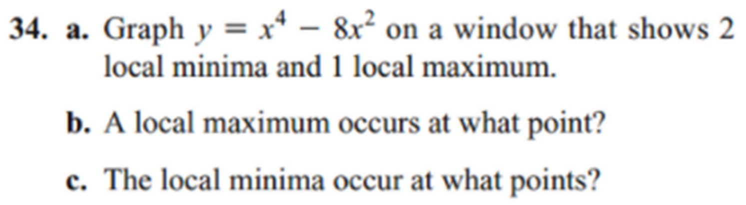
a. Complete the following table to see how increasing the interest rate affects the future value of this investment.

<table>
<tr><th>Rate, <i>r</i></th><th>Future Value, <i>S</i> ($)</th></tr>
<tr><td>0.00</td><td></td></tr>
<tr><td>0.05</td><td></td></tr>
<tr><td>0.10</td><td></td></tr>
<tr><td>0.15</td><td></td></tr>
<tr><td>0.20</td><td></td></tr>
</table>

b. Graph this function for $0 \le r \le 0.24$.

c. Use the table and/or graph to compare the future value if $r = 10\%$ and if $r = 20\%$. How much more money is earned at 20%?

d. Which interest rate, 10% or 20%, is more realistic for an investment?
</td>
</tr>
</table>

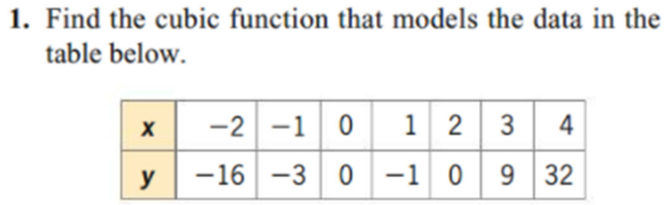
<table>
<tr>
<td>Find local minima, local maxima, absolute minima, and absolute maxima of polynomial functions.</td>
<td>


34. **a.** Graph $y = x^4 - 8x^2$ on a window that shows 2 local minima and 1 local maximum.

b. A local maximum occurs at what point?

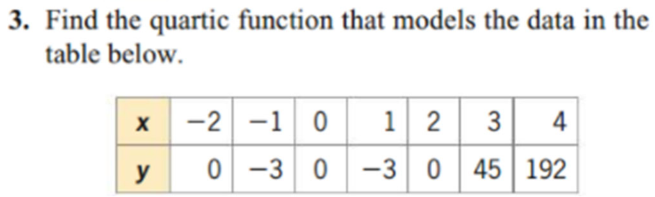
c. The local minima occur at what points?
</td>
</tr>
</table>

6.2 Modeling with Cubic and Quartic Functions

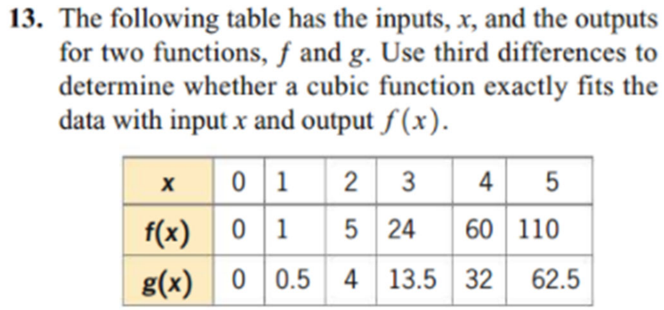
<table>
<tr>
<th>Learning Outcomes and Notes</th>
<th>Examples</th>
</tr>
<tr>
<td>Find cubic and quartic models for data in tables.</td>
<td>


1. Find the cubic function that models the data in the table below.

<table>
<tr><td>x</td><td>−2</td><td>−1</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>y</td><td>−16</td><td>−3</td><td>0</td><td>−1</td><td>0</td><td>9</td><td>32</td></tr>
</table>

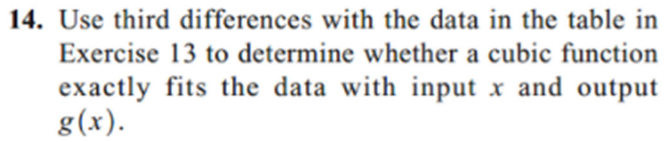


3. Find the quartic function that models the data in the table below.

<table>
<tr><td>x</td><td>−2</td><td>−1</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>y</td><td>0</td><td>−3</td><td>0</td><td>−3</td><td>0</td><td>45</td><td>192</td></tr>
</table>
</td>
</tr>
<tr>
<td>Determine the most appropriate model for data.</td>
<td>


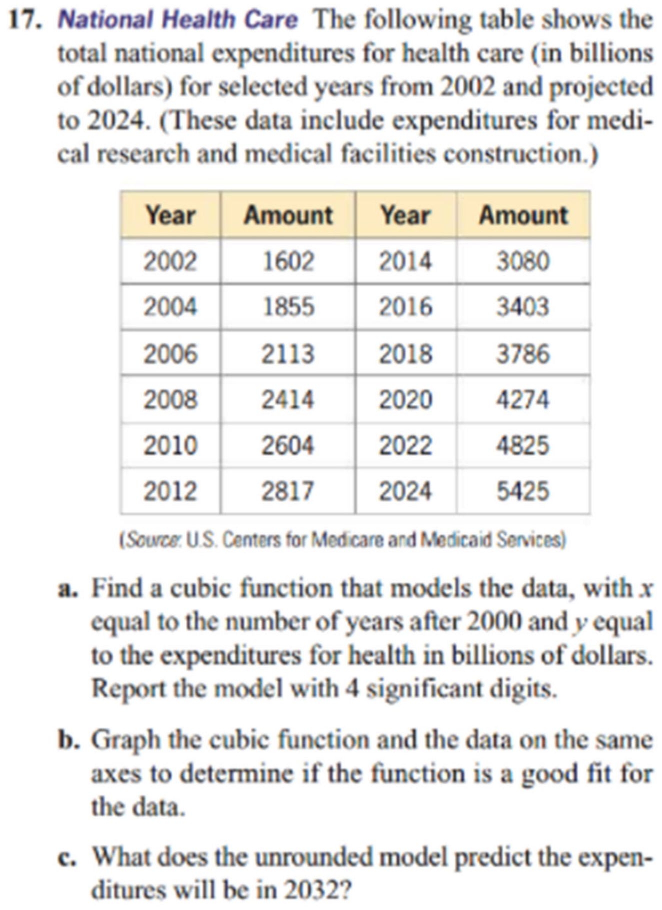
13. The following table has the inputs, x, and the outputs for two functions, f and g. Use third differences to determine whether a cubic function exactly fits the data with input x and output $f(x)$.

<table>
<tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td>f(x)</td><td>0</td><td>1</td><td>5</td><td>24</td><td>60</td><td>110</td></tr>
<tr><td>g(x)</td><td>0</td><td>0.5</td><td>4</td><td>13.5</td><td>32</td><td>62.5</td></tr>
</table>



14. Use third differences with the data in the table in Exercise 13 to determine whether a cubic function exactly fits the data with input x and output $g(x)$.
</td>
</tr>
</table>

Model and apply data with cubic functions.



17. ***National Health Care*** The following table shows the total national expenditures for health care (in billions of dollars) for selected years from 2002 and projected to 2024. (These data include expenditures for medical research and medical facilities construction.)

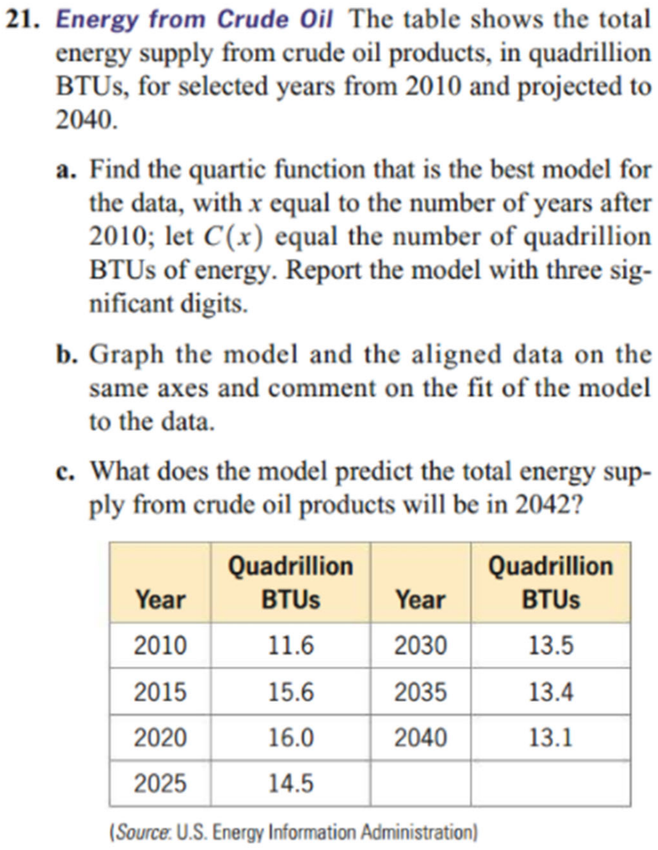
Year	Amount	Year	Amount
2002	1602	2014	3080
2004	1855	2016	3403
2006	2113	2018	3786
2008	2414	2020	4274
2010	2604	2022	4825
2012	2817	2024	5425

(*Source:* U.S. Centers for Medicare and Medicaid Services)

a. Find a cubic function that models the data, with x equal to the number of years after 2000 and y equal to the expenditures for health in billions of dollars. Report the model with 4 significant digits.

b. Graph the cubic function and the data on the same axes to determine if the function is a good fit for the data.

c. What does the unrounded model predict the expenditures will be in 2032?

<table>
<tr>
<td>Model and apply data with quartic functions.</td>
<td>


21. ***Energy from Crude Oil*** The table shows the total energy supply from crude oil products, in quadrillion BTUs, for selected years from 2010 and projected to 2040.

a. Find the quartic function that is the best model for the data, with x equal to the number of years after 2010; let $C(x)$ equal the number of quadrillion BTUs of energy. Report the model with three significant digits.

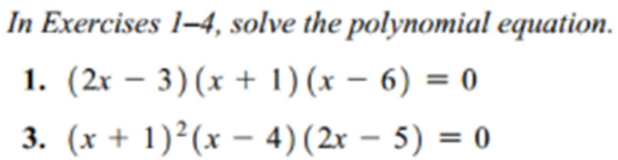
b. Graph the model and the aligned data on the same axes and comment on the fit of the model to the data.

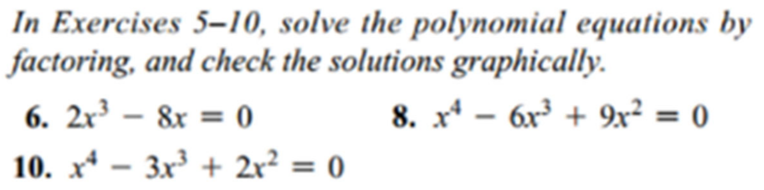
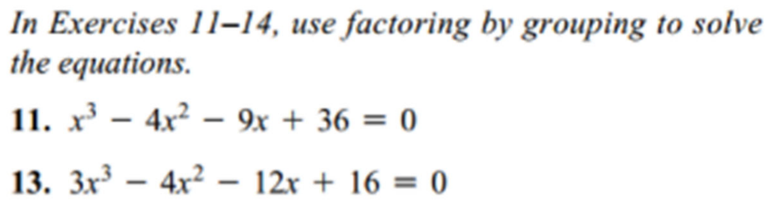
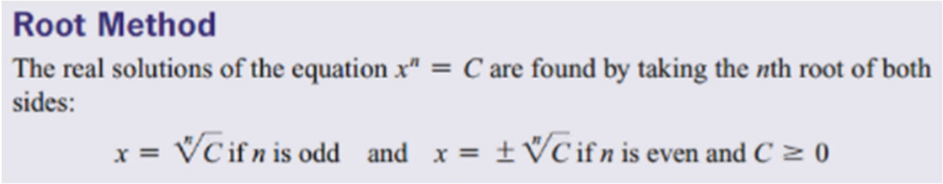
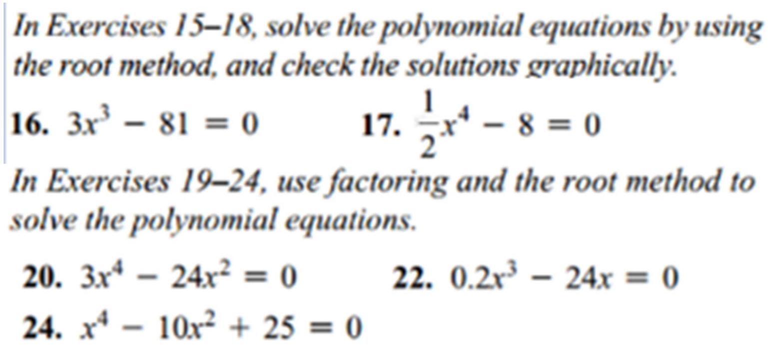
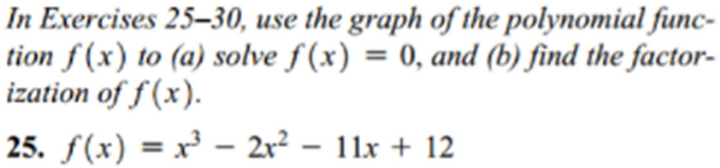
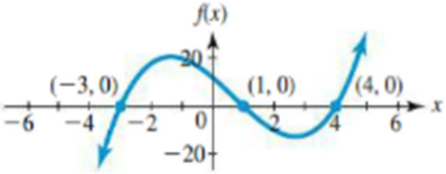
c. What does the model predict the total energy supply from crude oil products will be in 2042?

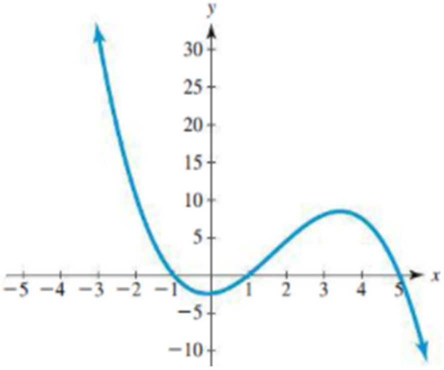
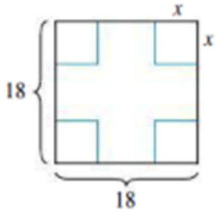
<table>
<tr><th>Year</th><th>Quadrillion BTUs</th><th>Year</th><th>Quadrillion BTUs</th></tr>
<tr><td>2010</td><td>11.6</td><td>2030</td><td>13.5</td></tr>
<tr><td>2015</td><td>15.6</td><td>2035</td><td>13.4</td></tr>
<tr><td>2020</td><td>16.0</td><td>2040</td><td>13.1</td></tr>
<tr><td>2025</td><td>14.5</td><td></td><td></td></tr>
</table>

(*Source:* U.S. Energy Information Administration)
</td>
</tr>
</table>

6.3 Solution of Polynomial Equations

Learning Outcomes and Notes	**Examples**
Solve factored polynomial equations.	 *In Exercises 1–4, solve the polynomial equation.* **1.** $(2x - 3)(x + 1)(x - 6) = 0$ **3.** $(x + 1)^2(x - 4)(2x - 5) = 0$

Solve polynomial equations using factoring.	 *In Exercises 5–10, solve the polynomial equations by factoring, and check the solutions graphically.* **6.** $2x^3 - 8x = 0$ **8.** $x^4 - 6x^3 + 9x^2 = 0$ **10.** $x^4 - 3x^3 + 2x^2 = 0$
Solve polynomial equations using factoring by grouping.	 *In Exercises 11–14, use factoring by grouping to solve the equations.* **11.** $x^3 - 4x^2 - 9x + 36 = 0$ **13.** $3x^3 - 4x^2 - 12x + 16 = 0$
Solve polynomial equations using the root method.  **Root Method** The real solutions of the equation $x^n = C$ are found by taking the nth root of both sides: $x = \sqrt[n]{C}$ if n is odd and $x = \pm\sqrt[n]{C}$ if n is even and $C \geq 0$	 *In Exercises 15–18, solve the polynomial equations by using the root method, and check the solutions graphically.* **16.** $3x^3 - 81 = 0$ **17.** $\frac{1}{2}x^4 - 8 = 0$ *In Exercises 19–24, use factoring and the root method to solve the polynomial equations.* **20.** $3x^4 - 24x^2 = 0$ **22.** $0.2x^3 - 24x = 0$ **24.** $x^4 - 10x^2 + 25 = 0$
Solve and factor polynomial equations using a graph.	 *In Exercises 25–30, use the graph of the polynomial function $f(x)$ to (a) solve $f(x) = 0$, and (b) find the factorization of $f(x)$.* **25.** $f(x) = x^3 - 2x^2 - 11x + 12$ 

	29. $y = -0.5x^3 + 2.5x^2 + 0.5x - 2.5$ 
Solve applications involving polynomial equations and factoring.	**41.** ***Profit*** The profit function for a product is given by $P(x) = -x^3 + 2x^2 + 400x - 400$, where x is the number of units produced and sold and P is in hundreds of dollars. Use factoring by grouping to find the numbers of units that will give a profit of \$40,000.
Answer questions involving the volume of a box.	**39.** ***Constructing a Box*** A box can be formed by cutting a square out of each corner of a piece of tin and folding the sides up. Suppose the piece of tin is 18 inches by 18 inches and each side of the square that is cut out has length x. **a.** Write an expression for the height of the box that is constructed.  **b.** Write an expression for the dimensions of the base of the box that is constructed. **c.** Use the formula $V = lwh$ to find an equation that represents the volume of the box.

Finding the Maximum Volume

A box is to be formed by cutting a square of x inches per side from each corner of a square piece of cardboard that is 24 inches on each side and folding up the sides. This will give a box whose height is x inches, with the length of each side of the bottom $2x$ less than the original length of the cardboard (see Figure 6.22). Using $V = lwh$, the volume of the box is given by

$$V = (24 - 2x)(24 - 2x)x$$

where x is the length of the side of the square that is cut out.

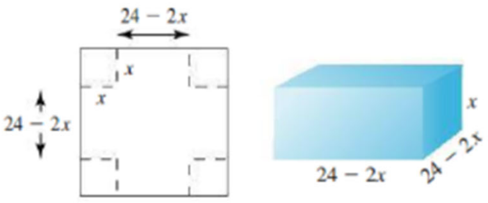


Figure 6.22

a. Use the factors of V to find the values of x that give volume 0.

b. Graph the function that gives the volume as a function of the side of the square that is cut out over an x-interval that includes the x-values found in part (a).

c. What input values make sense for this problem (that is, actually result in a box)?

SOLUTION

a. We use the zero-product property to solve the equation $0 = (24 - 2x)(24 - 2x)x$.

$$\begin{aligned} 0 &= (24 - 2x)(24 - 2x)x \\ x = 0 \quad &\text{or} \quad 24 - 2x = 0 \\ &\qquad -2x = -24 \\ x = 0 \quad &\text{or} \qquad x = 12 \end{aligned}$$

b. A graph of $V(x)$ from $x = -1$ to $x = 16$ is shown in Figure 6.23(a).

c. The box can exist only if each of the dimensions is positive, resulting in a volume that is positive. Thus, $24 - 2x > 0$, or $x < 12$, so the box can exist only if the value of x is greater than 0 and less than 12. Note that if squares 12 inches on a side are removed, no material remains to make a box, and no squares larger than 12 inches on a side can be removed from all four corners.

d. Use the equation that you constructed to find the values of x that make $V = 0$.

e. For which of these values of x does a box exist if squares of length x are cut out and the sides are folded up?